Taylor Series Error Bound

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This handout will first go over some common mistakes and show a simple problem to give an overview of how these problems work. Then I present the quiz solutions, which will go more in detail and give tips on how to solve these types of problems.

Common Mistakes

- The biggest mistake is that when you are finding the value M, it must be a bound on the absolute value of all the next power derivatives, *over the entire region.* This means that ****you cannot just evaluate the derivatives at the given point to find M****.
- Note that you are given the value of $|x-a|$ and $|y-a|$. That means that, for example, if given that $|x-1| \leq 0.1, |y-1| \leq 0.1$, in the error formula you would simply have $|E(x,y)| \leq \frac{1}{2}M(0.1+0.1)^2$. No other manipulation is necessary for this part.
- When dealing with $<, >$, note that the sign flips when dealing with the reciprocal. That is, if $y < 4$, it is **not** true that $\frac{1}{y} < 4$. For example, if you wanted to show that $\frac{1}{y} < 1$, you would need to show that $y > 1$. The same goes true for negation.
- M is a bound on the *absolute value* of the derivatives.
- Some people got confused on what the error bound formula was and confused it with Taylor series. See the last section for this in more detail.

Example

1. Let $f(x,y) = 2\cos(xy)$. Consider the linear approximation at the point $(0.1, 0.1)$. Show that on the region R given by $|x-0.1| \leq 0.1, |y-0.1| \leq 0.1$, we have the following bound on the error for this linear approximation:
$$|E(x,y)| \leq 0.0016$$
Solution: First, manipulate the region R to be more usable inequalities:
$$|x-0.1| \leq 0.1 \quad \rightarrow \quad -0.1 \leq x-0.1 \leq 0.1 \quad \rightarrow \quad \boxed{0 \leq x \leq 0.2}$$
$$|y-0.1| \leq 0.1 \quad \rightarrow \quad -0.1 \leq y-0.1 \leq 0.1 \quad \rightarrow \quad \boxed{0 \leq y \leq 0.2}$$
Next, use these to find bounds on the derivatives (using $|\cos(x)| \leq 1$)
$$|f_{xx}| = |-2y^2\cos(xy)| \leq 2y^2 \leq 2(0.2)^2 = 0.08$$
$$|f_{xy}| = |-2xy\cos(xy)| \leq 2xy \leq 2(0.2)^2 = 0.08$$
$$|f_{yy}| = |-2x^2\cos(xy)| \leq 2x^2 \leq 2(0.2)^2 = 0.08$$
We can therefore use $M = 0.08$. Plugging this all in gives
$$|E(x,y)| \leq \frac{1}{2}M(|x-0.1|+|y-0.1|)^2 = \frac{1}{2}(0.08)(0.1+0.1)^2 = (0.04)(0.04) = \boxed{0.0016}$$

Quiz 6 Question 1(c): Solutions

The solutions to both problems are typed out in full here. Both explanations are the same, just with different numbers.

Version 1: 10 A.M., 11 A.M., 1 P.M.

1. Let $f(x,y) = x\sqrt{y}$. You are given that

$$f_{xx} = 0 \qquad f_{xy} = \frac{1}{2\sqrt{y}} \qquad f_{yy} = \frac{-x}{4y^{3/2}}$$

Let (x,y) be a point such that

$$|x-1| \le 0.4 \quad \text{and} \quad |y-4| \le 0.4$$

Bound (estimate) the error of standard linear approximation of f at a point (x,y). Give your answer as a fully simplified fraction or decimal.

Solution: Note that since we are finding a linear approximation, the error formula is

$$|E(x,y)| \le \frac{1}{2}M(|x-1| + |y-4|)^2$$

The main goal is to find a suitable value for M. In this case, we need to find M such that

$$M \ge |f_{xx}| = 0 \qquad M \ge |f_{xy}| = \left|\frac{1}{2\sqrt{y}}\right| \qquad M \ge |f_{yy}| = \left|\frac{x}{4y^{3/2}}\right|$$

Note the presence of the absolute values. Now, in order to find M, we therefore need to find bounds on x and y. The only way to do so is to use the region given to us. Below, we split up the absolute value inequalities of x and y to extract bounds for each.

$$\begin{aligned} |x-1| \le 0.4 \quad &\to \quad 0.4 \le x - 1 \le 0.4 \\ &\quad\;\; 0.6 \le x \le 1.4 \quad \to \quad \boxed{x \le 1.4} \\ |y-4| \le 0.4 \quad &\to \quad 0.4 \le y - 4 \le 0.4 \\ &\quad\;\; 3.6 \le y \le 4.4 \quad \to \quad \boxed{3.6 \le y} \end{aligned}$$

Note that we will just need to know an upper bound for x. For y, we are going to need an upper bound for its reciprocal, which means we need a *lower* bound for y (in other words, $\frac{1}{y} \le \frac{1}{3.6}$). Below we use these inequalities to find M (note that $\sqrt{3.6} \ne 0.6$)

$$\begin{aligned} |f_{xy}| &= \left|\frac{1}{2\sqrt{y}}\right| \le \boxed{\frac{1}{2\sqrt{3.6}}} \\ |f_{yy}| &= \left|\frac{x}{4y^{3/2}}\right| \le \frac{1.4}{4y^{3/2}} \le \boxed{\frac{1.4}{4(3.6)^{3/2}}} \end{aligned}$$

It is at this point we have a choice. Usually, the question just asks for *an estimate*, not a specific or most optimal estimate. This means we can actually try to pick an easy M rather than spending effort to find a super small M. Usually good options tend to be $M = 1, 2, 5, 10$. Alternatively, if the question was asking to get below a specific error, then you would need to be more careful to find a "small as possible M". But usually the numbers would be more clean in that case (for example, if the derivatives were all pure sines and cosines, then you could bound all sines and cosines by 1).

In this case, we can simplify our bounds a bit to get easier numbers:

$$\frac{1}{2\sqrt{3.6}} \le 1$$

$$\frac{1.6}{4(3.6)^{3/2}} \le \frac{2}{4(3.6)^{3/2}} = \frac{1}{2(3.6)^{3/2}} \le 1$$

This means that $1 \ge |f_{xx}|, |f_{xy}|, |f_{yy}|$, so we can pick $M = 1$. Note again that, since we do not need to be optimal, I simply played with the numbers until I got a bound that was clearly less than 1. I of course could pick a smaller M (for example, $M = 1/2$ is another good option that is easy to see), and depending on the question you may need to, but in this case it is unnecessary. Instead, just pick an M that is easy to work with. **The important part is that you must show that your choice of M is a valid choice through the inequalities, even if it is obvious.** It is also good to pick M to be reasonably small.

Finally, we can use the error formula and get

$$\begin{aligned}
|E(x,y)| &\le \frac{1}{2}M(|x-1| + |y-4|)^2 \\
&\le \frac{1}{2}(|x-1| + |y-4|)^2 && \text{substituting in our } M \\
&\le \frac{1}{2}(0.4 + 0.4)^2 && \text{substituting in given values} \\
&\le \frac{1}{2}(0.64) = \boxed{0.32}
\end{aligned}$$

Version 2: 9 A.M., 12 P.M, 2 P.M., 3 P.M., 4 P.M.

1. Let $f(x,y) = \ln(x-2y)$. You are given that

$$f_{xx} = \frac{-1}{(x-2y)^2} \qquad f_{xy} = \frac{2}{(x-2y)^2} \qquad f_{yy} = \frac{-4}{(x-2y)^2}$$

Let (x,y) be a point such that

$$|x-3| \leq 0.1 \quad \text{and} \quad |y-1| \leq 0.1$$

Bound (estimate) the error of standard linear approximation of f at a point (x,y). Give your answer as a fully simplified fraction or decimal.

Solution: Note that since we are finding a linear approximation, the error formula is

$$|E(x,y)| \leq \frac{1}{2}M(|x-3| + |y-1|)^2$$

The main goal is to find a suitable value for M. In this case, we need to find M such that

$$M \geq |f_{xx}| = \left|\frac{-1}{(x-2y)^2}\right| \qquad M \geq |f_{xy}| = \left|\frac{2}{(x-2y)^2}\right| \qquad M \geq |f_{yy}| = \left|\frac{-4}{(x-2y)^2}\right|$$

Note the presence of the absolute values. Now, in order to find M, we therefore need to find bounds on x and y. The only way to do so is to use the region given to us. Note as well that the second derivatives are all very similar: in fact, it just comes down to getting an *upper* bound on $\frac{1}{(x-2y)^2}$, which comes down to getting a *lower* bound on $(x-2y)$. Below, we split up the absolute value inequalities of x and y to extract bounds for each.

$$\begin{aligned} |x-3| \leq 0.1 \quad &\to \quad -0.1 \leq x-3 \leq 0.1 \\ &\quad\;\; 2.9 \leq x \leq 3.1 \quad \to \quad \boxed{2.9 \leq x} \\ |y-1| \leq 0.1 \quad &\to \quad -0.1 \leq y-1 \leq 0.1 \\ &\quad\;\; 0.9 \leq y \leq 1.1 \quad \to \quad \boxed{y \leq 1.1} \end{aligned}$$

Note that we will just need to know a lower bound for x. For y, we are going to need a lower bound for its negative, which means we need an upper bound for y (in other words, $-y \geq -1.1$). These give us that

$$x - 2y \geq 2.9 - 2y \geq 2.9 - 2(1.1) = 0.7$$

Now we can use this to find a bound on M:

$$\begin{aligned} |f_{xx}| &= \left|\frac{-1}{(x-2y)^2}\right| = \frac{1}{(x-2y)^2} \leq \frac{1}{(0.7)^2} \\ |f_{xy}| &= \left|\frac{2}{(x-2y)^2}\right| = \frac{1}{(x-2y)^2} \leq \frac{2}{(0.7)^2} \\ |f_{yy}| &= \left|\frac{-4}{(x-2y)^2}\right| = \frac{4}{(x-2y)^2} \leq \frac{4}{(0.7)^2} \end{aligned}$$

It is at this point we have a choice. Usually, the question just asks for *an estimate*, not a specific or most optimal estimate. This means we can actually try to pick an easy M rather than spending effort to find a super small M. Usually good options tend to be $M = 1, 2, 5, 10$. Alternatively, if the question was asking to get below a specific error, then you would need to be more careful to find a "small as possible M". But usually the numbers would be more clean in that case (for example, if the derivatives were all pure sines and cosines, then you could bound all sines and cosines by 1).

In this case, we can simplify our bounds a bit to get easier numbers:

$$\frac{1}{(0.7)^2} \leq \frac{2}{(0.7)^2} \leq \frac{4}{(0.7)^2} \leq \frac{4}{(0.5)^2} = \frac{4}{0.25} = \boxed{8} \qquad \text{note: smaller on bottom = bigger fraction}$$

This means that $8 \geq |f_{xx}|, |f_{xy}|, |f_{yy}|$, so we can pick $M = 8$. Note again that, since we do not need to be optimal, I simply played with the numbers until I got a bound that was clearly less than 8. I of course could try to pick a smaller M, and depending on the question you may need to, but in this case it is unnecessary. Instead, just pick an M that is easy to work with (another good option is $M = 10$). **The important part is that you must show that your choice of M is a valid choice through the inequalities, even if it is obvious.** It is also good to pick M to be reasonably small.

Finally, we can use the error formula and get

$$\begin{aligned} |E(x,y)| &\leq \frac{1}{2}M(|x-3| + |y-1|)^2 \\ &\leq 4(|x-3| + |y-1|)^2 && \text{substituting in our } M \\ &\leq 4(0.1 + 0.1)^2 && \text{substituting in given values} \\ &\leq 4(0.04) = \boxed{0.16} \end{aligned}$$

A Note On The Error Formula

The error bound on $P_n(x,y)$ at the point (a,b) is given by the formula

$$|f(x,y) - P_n(x,y)| = |E(x,y)| \leq \boxed{\frac{1}{(n+1)!}M(|x-a|+|y-b|)^{n+1}}$$

where M is greater than the absolute values of all the $n+1$ partials of f. So, for example, the error bound on the linear approximation would be

$$\boxed{\frac{1}{2}M(|x-a|+|y-b|)^2}$$

where M is greater than all the absolute values of the second partials of f.

It is true that is related to the $n+1^{th}$ term of the Taylor polynomial of f. However, it is not related in the way you may think (Mainly, it is **not** just the evaluation of $n+1^{th}$ term at the point). For example, lets say you wanted to bound $P_2(x,y)$ at $(1,4)$. A common mistake is to then use the 3rd Taylor polynomial term as follows:

$$|E(x,y)| \leq \frac{1}{6}(f_{xxx}x^3 + 3f_{xxy}x^2y + 3f_{xyy}xy^2 + f_{yyy}y^3)$$

But this formula is **incorrect** and is misleading for a couple of reasons. First, there is actually an absolute value over the entire thing. Secondly, this would only be true if the point was $(0,0)$. At $(1,4)$, you would instead have

$$\begin{aligned}|E(x,y)| &\leq \frac{1}{6}|f_{xxx}(x-1)^3 + 3f_{xxy}(x-1)^2(y-4) + 3f_{xyy}(x-1)(y-4)^2 + f_{yyy}(y-4)^3| \\ &\leq \frac{1}{6}(f_{xxx}|x-1|^3 + 3f_{xxy}|x-1|^2|y-4| + 3f_{xyy}|x-1||y-4|^2 + f_{yyy}|y-4|^3)\end{aligned}$$

Finally, this makes it seem like you can just evaluate all third derivatives at $(1,4)$, but this is **not true**. Instead, you actually need to bound all the derivatives over the region. Typically, it is easier to bound all the derivatives at once. If we find some M such that $M \geq |f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|$, then we have

$$\begin{aligned}|E(x,y)| &\leq \frac{1}{6}(M|x-1|^3 + 3M|x-1|^2|y-4| + 3M|x-1||y-4|^2 + M|y-4|^3) \\ &\leq \frac{1}{6}M(|x-1|^3 + 3|x-1|^2|y-4| + 3|x-1||y-4|^2 + |y-4|^3) \\ &\leq \frac{1}{6}M(|x-1|+|y-4|)^3\end{aligned}$$

which is the same as the error formula from before. However, you could also bound each derivative separately using different bounds for each: If we find $M_1 \geq |f_{xxx}|, M_2 \geq |f_{xxy}|, M_3 \geq |f_{xyy}|, M_4 \geq |f_{yyy}|$, then we have

$$|E(x,y)| \leq \boxed{\frac{1}{6}(M_1|x-1|^3 + 3M_2|x-1|^2|y-4| + 3M_3|x-1||y-4|^2 + M_4|y-4|^3)}$$

which is then another valid formula one could use, and this expanded out form would similarly extend if one wanted to bound the error on $P_3(x,y), P_4(x,y)$, etc.