A Full-Proof Guide To Rational Function Integration

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It turns out that all the tricks for integrating rational functions (namely, 3) can be combined into a step-by-step process for integrating any rational function. Due to the properties of polynomials, this process ensures that any rational function can be broken down into easy-to-integrate parts, making any integrable solvable. This is not necessarily the fastest method, but it does eliminate all guesswork in these types of problems: If you understand where you are in the guide, you will know exactly what you need to use to integrate and what the results will be.

This guide was created by Corbin Stone Diaz after he did many integrals in his life and realized the process was fairly formulaic, contradicting the fact that many teachers seem to teach them as separate things.

This guide will be using the example problem below:

$$\int \frac{x^4+3x^3+4x^2+x}{x^3+3x^2+4x+2}\,dx$$

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1 Polynomial Long Division

Criterion: *The degree of the numerator is greater than or equal to the degree of the denominator.*

The leftover polynomial bits should be integrated into polynomial expressions.

$$\int \frac{x^4+3x^3+4x^2+x}{x^3+3x^2+4x+2}\,dx = \int \left(x + \frac{-x}{x^3+3x^2+4x+2}\right)dx = \frac{1}{2}x^2 + \int \frac{-x}{x^3+3x^2+4x+2}\,dx$$

What is left to integrate: *A rational function in which the degree of the numerator is strictly less than the denominator.*

2 Partial Fraction Decomposition

Criterion: *The denominator can be factorized.*

Note that every polynomial can be factorized as either linear terms ($2x-3$, $x+1$, x, etc.) or quadratic polynomials with no real roots (x^2+1, x^2+x+1, etc.). Partial Fraction decomposition will ensure the rational function is split into these more manageable parts. At this point, all terms of the form “constant / linear” become a logarithmic expression when integrated.

$$\int \frac{-x}{x^3+3x^2+4x+2}\,dx = \int \left(\frac{1}{x+1} + \frac{-x-2}{x^2+2x+2}\right)dx = \ln|x+1| + \int \frac{-x-2}{x^2+2x+2}\,dx$$

End Result/What is left to integrate: *Rational functions in which the denominator is an unfactorable quadratic, and the numerator is either a linear term or a constant.*

3 “Derivative On Top!”

Criterion: *There is a term of the form ’linear / quadratic’, in which the quadratic is unfactorable.*

This whole step can be thought of as an edge case to handle one specific type of term. You may never need to perform this step. In order to integrate this rational function, it must be rewritten so that the numerator is a constant multiple of the derivative of the denominator. You will then have some left over bits that can be integrated in Step (4). This is the trickiest part because it is not ’taught’, there is no specific rule or formula that goes with this, mostly because it is rare. However, this step is fairly straightforward once you understand it. All of these terms become a logarithmic expression (now with a quadratic inside) when integrated.

$$\int \frac{-x-2}{x^2+2x+2}\,dx = \int -\frac{1}{2}\left(\frac{2x+2}{x^2+2x+2}\right)dx + \int \frac{-1}{x^2+2x+2}\,dx$$

$$= -\frac{1}{2}\ln|x^2+2x+2| + \int \frac{-1}{x^2+2x+2}\,dx$$

End Result/What is left to integrate: *Rational functions in which the denominator is an unfactorable quadratic, and the numerator is strictly a constant.*

4 Complete the square

Criterion: *There is a term of the form “constant / quadratic”, in which the quadratic is unfactorable.*

All of these terms can be solved using complete the square and become an arctangent expression when integrated.

$$\int \frac{-1}{x^2+2x+2}\,dx = \int \frac{-1}{(x+1)^2+1}\,dx = -\arctan(x+1)+C$$

The final answer is then:

$$\int \frac{x^4++3x^3+4x^2+x}{x^3+3x^2+4x+2}\,dx = \boxed{\frac{1}{2}x^2+\ln|x+1|-\frac{1}{2}\ln|x^2+2x+2|-\arctan(x+1)+C}$$