Calculus Bible

Version 2.4

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DISCLAIMER: This Bible should not be blindly memorized, but instead it is highly recommended to understand the concepts and derivations contained within.

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1 Limits

1.1 Common Limits

All of these can be proven with known techniques, e.g., logarithms and L'Hopital's.

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0 \qquad \lim_{x\to 0}\frac{\arcsin x}{x}=1$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab} \qquad \lim_{x\to\infty}x^{\frac{1}{x}}=1 \qquad \lim_{x\to 0^+}x^x=1$$

$$\lim_{x\to\infty}\sqrt[n]{ax^n+bx^{n-1}+\ldots}-\sqrt[n]{ax^n+cx^{n-1}+\ldots} = \frac{b-c}{na^{(n-1)/n}}$$

$$\lim_{x\to\infty}\sqrt[n]{x^n+bx^{n-1}+\ldots}-\sqrt[m]{x^m+cx^{m-1}+\ldots} = \frac{b}{n}-\frac{c}{m}$$

1.2 L'Hopital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

1.3 ϵ-δ Definition

Let $f(x)$ be a function defined on an interval that contains $x=a$ ($f(a)$ does not have to be defined).

$$\lim_{x\to a}f(x)=L$$

if for every $\epsilon>0$ there exists $\delta>0$ such that for all x,

$$|f(x)-L|<\epsilon \quad \text{whenever} \quad 0<|x-a|<\delta$$

1.4 Stirling's Approximation

$$n!\approx\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$$

This approximation can be used as a substitution for limits where $n\to\infty$.

1.5 Riemann Sum Limits

Test writers often like to disguise integrals as limits of Riemann sums. For integrable function $f(x)$, and real numbers a,k,l where $k,l>0$:

$$\lim_{n\to\infty}\sum_{i=0}^{kn}f\left(\left(\frac{l}{n}\right)i+a\right)\left(\frac{l}{n}\right)=\int_a^{kl+a}f(x)\,dx$$

Note: Subtle changes, such as replacing with "$kn\pm 1$", "$i=1$", or "$i\pm 1$" do not affect the above formula.

1.6 Other Limit Simplification Techniques

1.6.1 Using Logs

Most limit problems of the form

$$\lim_{x\to a}(f(x))^{g(x)}=L$$

can be simplified by taking the $\ln x$ of both sides:

$$\lim_{x\to a}g(x)\ln f(x)=\ln L$$

Specifically, if $\lim_{x\to a}f(x)=0$ and $\lim_{x\to a}g(x)=\infty$, then

$$\lim_{x\to a}(1+f(x))^{g(x)}=e^{\left(\lim_{x\to a}f(x)g(x)\right)}$$

1.6.2 Substitution

Note that if $x = \frac{1}{t}$, then $x \to 0^+$ and $t \to \infty$ are equivalent. This can be used in order to make limits of the indeterminate form "$0 \cdot \infty$" suitable for L'Hopital's. For example:

$$\lim_{x \to 0^+} x \ln x = \lim_{t \to \infty} \frac{\ln 1/t}{t} \xrightarrow{L'H} \lim_{t \to \infty} -\frac{1}{t} = 0$$

1.6.3 Series Expansion

For limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ with one or multiple trig functions mixed with polynomials, it may be more suitable to use a series expansion rather than L'Hopital's. For example:

$$\lim_{x \to 0} \frac{\arctan x - x + \frac{x^3}{3}}{x^5} = \lim_{x \to 0} \frac{(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots) - x + \frac{x^3}{3}}{x^5} = \lim_{x \to 0} \left(\frac{1}{5} - \frac{x^2}{7} + \ldots \right) = \frac{1}{5}$$

Series expansion is widely applicable and makes quick work of limits which would be harder or even impossible to solve using L'Hopital's, making it almost always the superior alternative.

1.7 Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be continuous functions on $[a, b]$ (except possibly at $a \leq c \leq b$), and for all $x \in [a, b]$,

$$f(x) \leq h(x) \leq g(x)$$

If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$, then $\lim_{x \to c} h(x) = L$. This can be used to solve problems where $h(c)$ may be harder to evaluate than $f(c)$ and $g(c)$. For example, given that

$$\lim_{x \to 0} -|x| \leq \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} |x|$$

we can evaluate the left and right limits as 0, allowing us to conclude that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

1.8 Summary of Indeterminate Forms and Techniques

For each indeterminate form, most limits can be solved using one or more of the simple methods mentioned above:

Indeterminate Form	Common Initial Step
$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty}$	L'Hopital's or Series Expansion
$1^\infty,\ 0^0,\ \infty^0$	Take $\ln x$ of both sides
$0 \cdot \infty$	Substitution
$\infty - \infty$	Multiply by Conjugate

2 Derivatives

2.1 Common Derivatives

2.1.1 Exponential/Logarithm Functions

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a) \qquad \frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln(a)}$$

2.1.2 Trig Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

2.1.3 Inverse Trig Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

2.1.4 Hyperbolic Trig Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

2.1.5 Inverse Hyperbolic Trig Functions

$$\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}} \qquad \frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(\operatorname{arcsech} x) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arccsch} x) = -\frac{1}{|x|\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{arccoth} x) = \frac{1}{1-x^2}$$

2.2 Basic Formulas

2.2.1 Power Rule

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

2.2.2 Product Rule and Generalizations

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

This can be generalized to more than two functions:

$$\frac{d}{dx}(f(x)g(x)h(x)) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$\vdots$$

It can further be generalized to higher order derivatives using the **Leibniz Formula** or **General Leibniz Rule**. If $h(x) = f(x)g(x)$, then:

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$$

This formula is akin to the binomial theorem. However, it is rarely needed.

2.2.3 Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

A mnemonic to help you remember is "Lo D-Hi minus Hi D-Lo over LoLo."

2.2.4 Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

This is sometimes written in the form:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This can be remembered by noting that the du's "cancel" and is useful when working with parametric equations.

2.3 Even/Odd Functions for Derivatives and Trig Cycles

The derivative of an even function is always an odd function and the derivative of an odd function is always an even function.

In particular, $\sin x$ and $\cos x$ have derivatives that repeat every 4th iteration:

$$\begin{aligned} (\sin x)' &= \cos x \\ (\sin x)'' &= -\sin x \\ (\sin x)''' &= -\cos x \\ (\sin x)'''' &= \sin x \end{aligned} \qquad\qquad \begin{aligned} (\cos x)' &= -\sin x \\ (\cos x)'' &= -\cos x \\ (\cos x)''' &= \sin x \\ (\cos x)'''' &= \cos x \end{aligned}$$

2.4 Implicit Differentiation

For implicit functions $f(x, y)$, rather than trying to get it in the form $f(x) = y$, it is much easier to just take the derivative of both sides. For example, both methods below find the derivative of a circle, but the second method, which uses implicit differentiation, is much faster and easier:

$$x^2 + y^2 = r^2 \implies y = \pm\sqrt{r^2 - x^2} \implies y' = -\frac{2x}{2(\pm\sqrt{r^2 - x^2})} = -\frac{x}{y}$$

$$x^2 + y^2 = r^2 \implies 2x + 2yy' = 0 \implies y' = -\frac{x}{y}$$

Note that since y is a function of x, when taking the derivative of both sides with respect to x, the chain rule must be used. Also make sure to double check that the point (x_0, y_0) that you are finding the derivative of at actually exists on the implicit curve $f(x, y)$ before computing.

Specifically, we have that:

$$\text{Given the curve } x^n + y^n = A: \quad y' = -\left(\frac{x}{y}\right)^{n-1}, \quad y'' = -\frac{A(n-1)x^{n-2}}{y^{2n-1}}$$

2.4.1 Logarithmic Differentiation

For functions where both the base and the exponent include variables, sometimes it is useful to take the natural log of the functions and then perform implicit differentiation. For example:

$$y = x^x \implies \ln(y) = x\ln(x) \implies \frac{y'}{y} = 1 + \ln(x) \implies y' = x^x(1 + \ln(x))$$

In a more generalized sense,

$$\frac{d}{dx}\left(f(x)^{g(x)}\right) = f(x)^{g(x)}\left(g(x)\frac{f'(x)}{f(x)} + g'(x)\ln(f(x))\right)$$

Logarithmic differentation is also useful in pre-factored polynomials and rational functions. For example:

$$f(x) = \frac{(x-1)(x+3)^2}{(x-2)^3} \implies \ln f(x) = \ln(x-1) + 2\ln(x+3) - 3\ln(x-2) \implies f'(x) = f(x)\left(\frac{1}{x-1} + \frac{2}{x+3} - \frac{3}{x-2}\right)$$

In a generalized sense, for roots r_i and exponents e_i,

$$\frac{d}{dx}\left[\prod_{i=0}^{n}(x-r_i)^{e_i}\right] = \left(\prod_{i=0}^{n}(x-r_i)^{e_i}\right)\left(\sum_{i=0}^{n}\frac{e_i}{(x-r_i)}\right)$$

A closer look of this method reveals it is also just another representation of the product rule. This same technique can be used on more complex functions as well. Essentially, rather than using multiple iterations of the product and quotient rule, logarithmic differentiation allows us to take the derivative much more easily:

$$f(x) = \frac{2e^{-x}\sin(x)}{5x+1} \implies \ln f(x) = \ln 2 - x + \ln(\sin x) - \ln(5x+1) \implies f'(x) = f(x)\left(-1 + \cot x - \frac{5}{5x+1}\right)$$

2.4.2 Using Partial Derivatives

For implicit curves in the form $f(x, y) = 0$, rather than using implicit differentiation it may be easier to use partial derivatives to find the derivative. In simplest terms, the partial derivative with respect to x is the same as a regular derivative, just pretending all other variables (y) are constants. Considering the total derivative of $f(x, y)$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \implies \frac{dy}{dx} = -\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} \implies -\left(\frac{\text{pretend } y \text{ is constant}}{\text{pretend } x \text{ is constant}}\right)$$

For example, both methods below find the derivative of the implicit curve, but the second method, which uses partial derivatives, is much faster and easier:

$$x^2+xy+2xy^2+3 = 0 \implies 2x+y+x\frac{dy}{dx}+2y^2+4xy\frac{dy}{dx} = 0 \implies (x+4xy)\frac{dy}{dx} = -(2x+y+2y^2) \implies \frac{dy}{dx} = -\frac{2x+y+2y^2}{x+4xy}$$

$$\frac{\partial}{\partial x}(x^2+xy+2xy^2+3) = 2x+y+2y^2, \quad \frac{\partial}{\partial y}(x^2+xy+2xy^2+3) = x+4xy \implies \frac{dy}{dx} = -\frac{2x+y+2y^2}{x+4xy}$$

2.5 Derivatives of Inverse Functions

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

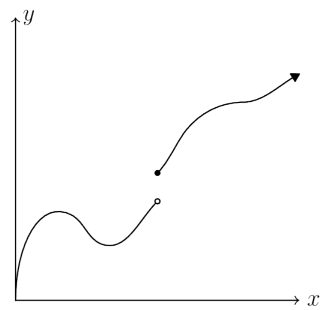
2.6 Analyzing Functions

2.6.1 Continuity and Differentiability

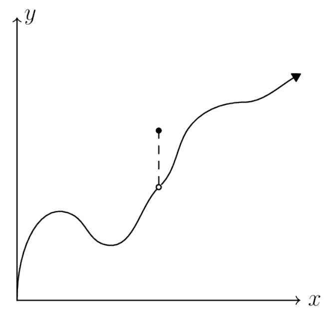
A function $f(x)$ is continuous at a provided all three of the following are true:

1. $f(a)$ exists 2. $\lim_{x \to a} f(x)$ exists 3. $\lim_{x \to a} f(x) = f(a)$

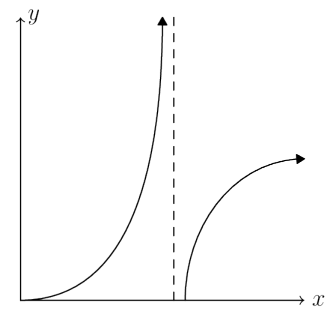
There are three main types of discontinuities that are asked about:



Jump Discontinuity



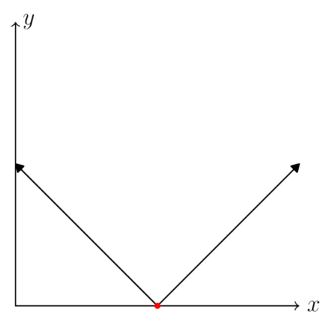
Removable Discontinuity



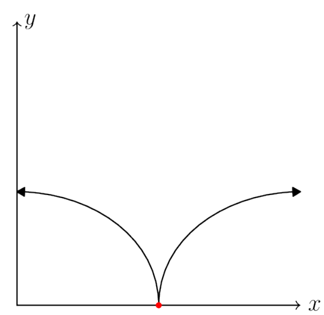
Infinite Discontinuity

Infinite discontinuities are a type of **essential discontinuity**, which are any types of discontinuities in which the limit is DNE or infinite. Another example is $y = \sin \frac{1}{x}$, which is discontinuous at 0.

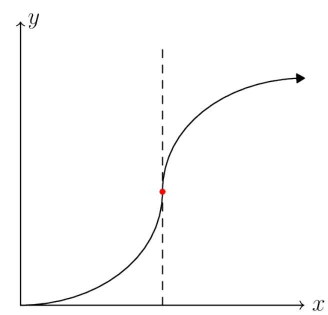
If a function f is differentiable, it must be continuous. However, just because a function f is continuous, does not make it differentiable. Below are the three main cases in which a point is continuous but not differentiable:



Bend



Cusp



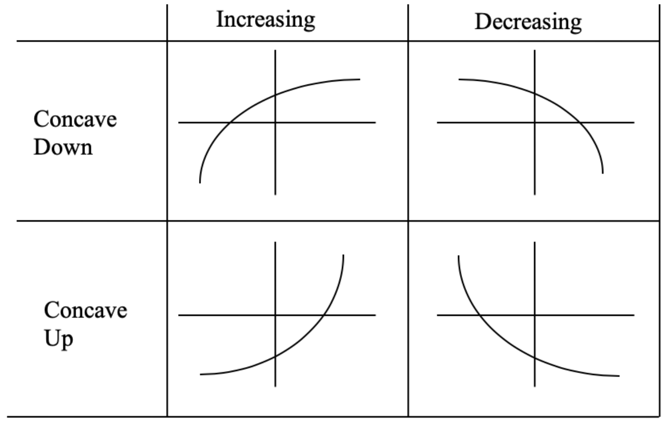
Vertical Tangent

When evaluating if a piecewise function is differentiable, it must also be continuous, and thus you must check both that the *value* of the function is the same at both ends and that the *derivative* of the function is the same at both ends.

2.6.2 Increasing, Decreasing, Concavity

For a function $f(x)$, $f'(x)$ represents the *instantaneous* rate of change of y with respect to x, which also implies that $f'(a)$ is the slope of the tangent line to $f(x)$ at $x = a$. This means that when $f'(x) > 0$, the function is increasing and when $f'(x) < 0$, the function is decreasing.

Similarly, $f''(x)$ denotes the double derivative of $f(x)$, or the *instantaneous* rate of change of the derivative with respect to x. The more common term for this is a function's concavity. When $f''(x) > 0$, the function is concave up and when $f''(x) < 0$, the function is concave down.



2.6.3 Critical Points and Inflection Points

We say that $x = c$ is a critical point of $f(x)$ if

1. $f(c)$ exists 2. $f'(c) = 0$ or DNE

At a critical point, a function can either have a

1. Maximum: $f'(x)$ changes from positive to negative or $f''(x)$ is negative.
2. Minimum: $f'(x)$ changes from negative to positive or $f''(x)$ is positive.

Formally, a point $x = c$ is called an inflection point if $f(x)$ is continuous and changes concavity at that point ($f''(x)$ change sign). Because of this, a vertical tangent is classified as both a critical and inflection point. In this vein, if a point $x = c$ has $f'(c) = 0$ and is an inflection point, we call $x = c$ a saddle point.

2.6.4 Extreme Value Theorem (EVT)

Suppose that $f(x)$ is continuous on the interval $[a, b]$. Then, there exist two numbers c, d in the closed interval $[a, b]$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

An extremum (maximum/minimum) can be called a global/absolute extremum if it is the maximum/minimum value of $f(x)$ over the entire domain that it is defined on. On the other hand, all extrema fall under the classification of local/relative extrema. When EVT does not apply (the interval is open and/or there are discontinuities), there can still be absolute extrema although it is not guaranteed.

2.6.5 Intermediate Value Theorem (IVT)

Let $f(x)$ be continuous on the closed interval $[a, b]$ such that $f(a) \neq f(b)$. Then for all $f(a) < k < f(b)$, there exists some $a < c < b$ such that $f(c) = k$.

Not all differentiable functions f have a continuous derivative. However, it turns out that the intermediate value property still holds for every derivative via **Darboux's theorem**. Additionally, any function that follows the intermediate value property is called a **Darboux function**.

2.6.6 Mean Value Theorem (MVT)

Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Specifically, the value c guaranteed by MVT for a polynomial of degree of at most 2 is always $(a + b)/2$.

2.6.7 Extended Mean Value Theorem (EMVT)

Let $f(x)$ and $g(x)$ be functions continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) such that $g'(x) \neq 0$ for all $a < x < b$. Then, there exists a number c such that $a < c < b$ and

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

This theorem however is rarely needed.

2.6.8 Rolle's Theorem

Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, there is a number c such that $a < c < b$ and $f'(c) = 0$. This can also be seen as a specific case of the Mean Value Theorem.

2.6.9 Higher Order Derivatives

Function	Name
$f(x)$	Position (Displacement)
$f'(x)$	Velocity
$f''(x)$	Acceleration
$f^{(3)}(x)$	Jerk
$f^{(4)}(x)$	Snap (Jounce)
$f^{(5)}(x)$	Crackle (Flounce)
$f^{(6)}(x)$	Pop (Pounce)

2.7 Approximations

2.7.1 Newton's Method

Newton's Method for approximating the roots of a function is recursive and involves finding tangent lines from successive approximations. Starting from an initial approximation of x_0, for $n > 0$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

2.7.2 Linear Approximations

Sometimes called approximation by differentials, $f(a + \Delta x)$ can be approximated using the equation for the tangent line at $f(a)$:

$$f(a + \Delta x) \approx f'(a)\Delta x + f(a)$$

3 Integration

3.1 Common Integrals

Some of the simpler integrals are just the reverse of the derivatives shown above and have been omitted. The only one with a slight "change" that should be mentioned is that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.

3.1.1 Trig Functions

$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$

$\int \cot x\,dx = \ln|\sin x| + C = -\ln|\csc x| + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

$$\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

$$\int \csc^3 x\,dx = -\frac{1}{2}(\csc x \cot x + \ln|\csc x + \cot x|) + C$$

3.1.2 Exponential/Logarithm Functions

$\int \ln x\,dx = x\ln x - x + C$

$\int xe^x\,dx = (x-1)e^x + C$

$\int \dfrac{1}{x\ln x}\,dx = \ln|\ln x| + C$

$\int \dfrac{\ln x}{x}\,dx = \dfrac{1}{2}(\ln x)^2 + C$

$\int e^{ax}\sin(bx)\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx)) + C$

$\int x^n \ln x\,dx = \dfrac{x^{n+1}}{n+1}\ln x - \dfrac{x^{n+1}}{(n+1)^2} + C$

$\int e^{ax}\cos(bx)\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx)) + C$

3.1.3 Inverse Trig Functions

$\int \dfrac{dx}{\sqrt{a^2-x^2}} = \arcsin\dfrac{x}{a} + C$

$\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a} + C$

$\int \dfrac{dx}{x\sqrt{x^2-a^2}} = \dfrac{1}{a}\,\mathrm{arcsec}\,\dfrac{x}{a} + C$

$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C$

$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + C$

$\int \arctan x\,dx = x\arctan x - \dfrac{1}{2}\ln(1+x^2) + C$

3.1.4 Miscellaneous

$\int \sqrt{x^2 \pm a^2}\,dx = \dfrac{a^2}{2}\ln|x + \sqrt{x^2 \pm a^2}| + \dfrac{x}{2}\sqrt{x^2 \pm a^2} + C$

$\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$

$\int \dfrac{dx}{(ax+b)(cx+d)} = \dfrac{1}{ad-bc}\ln\left|\dfrac{ax+b}{cx+d}\right| + C$

$\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ln|ax+b| + C$

3.2 First Fundamental Theorem of Calculus

If $f(x)$ is continuous on the closed interval $[a,b]$, then for $F(x) = \int_a^x f(t)\,dt$,

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}F(x) = f(x)$$

3.3 Second Fundamental Theorem of Calculus

If $f(x)$ is continuous on the closed interval $[a,b]$ and $F(x)$ is the antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

3.4 Mean Value Theorem for Integrals

Let $f(x)$ be continuous function on the interval $[a, b]$. Then there exists a number c such that $a < c < b$ and

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

This means that there exists a value c on the interval such that $f(c)$ is equal to the average value of $f(x)$ on that interval.

3.5 Integral Approximations

Let $f(x)$ be defined on an interval $[a, b]$ where $a = x_0 < x_1 < \ldots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$. The four main types of Riemann sums are defined as follows:

- **Left Riemann Sum:**

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

- **Right Riemann Sum:**

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

- **Midpoint Riemann Sum:**

$$\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$$

- **Trapezoidal Sum:**

$$\sum_{i=1}^{n} \left(\frac{f(x_{i-1})+f(x_i)}{2}\right)\Delta x_i$$

Note that the trapezoidal sum is just the average of the right and left Riemann sums.

- **Simpson's Rule:** Requires an even number of subdivisions and is exact for linear, quadratic, and cubic polynomials:

$$\int_a^b f(x)\, dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)]$$

If S is the Simpson's approximation using $2n$ subintervals, and M and T are the Midpoint and Trapezoid approximations using n subintervals, respectively, then

$$S = \frac{2}{3}M + \frac{1}{3}T$$

3.5.1 Accuracy

You can tell if a sum is an over or under approximation by the following chart:

Method	Underestimate	Overestimate
Left Riemann Sum	$dy/dx > 0$	$dy/dx < 0$
Right Riemann Sum	$dy/dx < 0$	$dy/dx > 0$
Trapezoidal Sum	$d^2y/dx^2 < 0$	$d^2y/dx^2 > 0$
Midpoint Sum	$d^2y/dx^2 > 0$	$d^2y/dx^2 < 0$

When approximating over the interval $[a, b]$ with n subintervals, the main error bound formulas are:

- Midpoint Rule $(\max(|f''(x)|) = K)$:

$$E_M \leq \left|\frac{K(b-a)^3}{24n^2}\right|$$

- Trapezoidal Rule $(\max(|f''(x)|) = K)$:

$$E_M \leq \left|\frac{K(b-a)^3}{12n^2}\right|$$

- Simpson's Rule ($\max(|f^{(4)}(x)|) = K$) :
$$E_M \leq \left| \frac{K(b-a)^5}{180n^4} \right|$$

These are rarely used, but allow you to determine the minimum number of subdivisions needed to approximate an integral given an error bound E_M.

3.6 Integration Techniques

This section heavily derives from Connor Gordon's handout on Integration Techniques, but is not a substitute for it. If you have the time, it is greatly encouraged to go through his guide as it is more in-depth and contains many practice problems.

3.6.1 U-Substitution

By using the substitution $u = g(x)$ and $du = g'(x)\,dx$, you can essentially apply a "reverse chain rule".

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

When applying this trick with definite integrals, the bounds must be changed from $[a, b]$ to $[g(a), g(b)]$.

Sometimes, you can apply a "backwards" U-substitution where $x = g(u)$ and $dx = g'(u)\,du$. For example, consider the following integral where we use the substitution $x = u^2 - 1$ and $dx = 2u\,du$.

$$\int_0^3 \sqrt{1+x}\,dx = \int_1^2 u(2u)\,du = \left[\frac{2}{3}u^3\right]_1^2 = \boxed{\frac{14}{3}}$$

Another less common way to express U-substitution is the Riemann-Stieltjes integral, which is defined as:

$$\int_{x=a}^{b} f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx$$

Note that the bounds refer to values of x, not $g(x)$. For example:

$$\int_1^\infty e^{-x^2}\,d(x^2+1) = \int_1^\infty e^{-x^2} 2x\,dx = \int_1^\infty e^{-u}\,du = \frac{1}{e}$$

3.6.2 Even/Odd Functions for Integrals and Trig Cycles

If $f(x)$ is an odd function and a is a real number,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx = 0$$

If $f(x)$ is an even function and a is a real number,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx = 2\int_0^a f(x)\,dx$$

The integral of an even function is always an odd function and the integral of an odd function is always and even function. The functions $\sin x$ and $\cos x$ as repeat every 4th iteration of integration similar to derivatives but backwards.

3.6.3 Improper Integrals

Improper integrals of the first kind can be written as a limit by using a dummy variable. If the limit exists, the integral converges; if not, the integral diverges.

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty} \int_0^b f(x)\,dx$$

Improper integrals of the second kind is a definite integral taken over an interval that contains an infinite discontinuity (i.e. asymptote). These limits are generally split into two limits and each part is evaluated separately. For example, although the following integral is odd, both of its limits do not exist.

$$\int_{-1}^{1} \frac{1}{x}\,dx = \lim_{b\to 0}\left[\int_{-1}^{b} \frac{1}{x}\,dx + \int_{b}^{1} \frac{1}{x}\,dx\right] = \lim_{b\to 0}\Big[\ln|x|\Big]_{-1}^{b} + \lim_{b\to 0}\Big[\ln|x|\Big]_{b}^{1}$$

3.6.4 Rational Functions

Most rational functions can be simplified through division and partial fraction decomposition, after which the integrals are pretty easy to compute. For example,

$$\int \frac{x^4+3x^3+4x^2+4x+4}{x^3+x^2+x+1}\,dx = \int x+2+\frac{1}{x+1}+\frac{1}{x^2+1}\,dx = \boxed{\frac{x^2}{2}+2x+\ln|x+1|+\arctan(x)+C}$$

For quadratics that have no real roots and cannot be factored, you cannot use a method of partial fractions. First, separate factors of x from the integral and put it in the form of the derivative of the denominator, then for the remainder complete the square to get in the form of an $\arctan x$:

$$\int \frac{x+2}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{1}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + \int \frac{1}{(x+1)^2+4}\,dx =$$

$$\boxed{\frac{1}{2}\ln|x^2+2x+5| + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right)+C}$$

In total, every rational function can be integrated using the following method:

1. First reduce through long division until the numerator has a smaller degree than the denominator (integrate the polynomial part as normal)
2. Factor the denominator and separate into partial fractions. Integrate all fractions with a linear term on the bottom as an $\ln|\ldots|$ term.
3. For remaining quadratic denominators with non-real roots, eliminate x terms in the numerator by manipulating it so the numerator is the derivative of the denominator, then u-sub the denominator to integrate as an $\ln|\ldots|$ term.
4. For remaining quadratic denominators bits with a constant on the top, complete the square to integrate as an $\arctan(\ldots)$ term.

For improper integrals of the first kind, the integral converges if the highest degree in the denominator is greater than 1. For improper integrals of the second kind involving an asymptote at $x=k$, write the numerator and denominator as polynomials in $(x-k)$. After cancelling common factors, the integral converges converges if and only if the lowest degree term in the denominator is less than 1.

3.6.5 Adding and Subtracting

Some integrals, such as those involving rational functions of $\sin(x)$ and $\cos(x)$ or e^x and e^{-x}, can be evaluated by adding and subtracting a constant or other term to set up for a u-substitution. For example,

$$\int \frac{\sin(x)}{\sin(x)+2\cos(x)}\,dx = \int \frac{\sin(x)}{\sin(x)+2\cos(x)} + c - c\,dx = \int \frac{(1+c)\sin(x)+2c\cos(x)}{\sin(x)+2\cos(x)} - c\,dx$$

To make the numerator a multiple of the derivative of the denominator, we need $(1+c)\sin(x)+2c\cos(x) = k(\cos(x) - 2\sin(x))$. Matching coefficients, we get $1+c=-2k$ and $2c=k$, which give $c=-\frac{1}{5}$ and $k=-\frac{2}{5}$.

$$\int \frac{\sin(x)}{\sin(x)+2\cos(x)}\,dx = \int \frac{-\frac{2}{5}(\cos(x)-2\sin(x))}{\sin(x)+2\cos(x)} + \frac{1}{5}\,dx = \boxed{\frac{x}{5}-\frac{2}{5}\ln|\sin(x)+2\cos(x)|+C}$$

3.6.6 Multiplying the Numerator and Denominator

For certain integrals, it can often be helpful to multiply the numerator and denominator by a common value. For example, the integral

$$\int \frac{1+x+x^2+x^4}{6x+4x^2+3x^3+2x^5}\,dx$$

seems hard to deal with by using partial fraction decomposition. Instead, we can multiply the numerator and denominator by x and then perform a u-substitution for $u = 6x^2+4x^3+3x^4+2x^6$.

$$\int \frac{1+x+x^2+x^4}{6x+4x^2+3x^3+2x^5}\,dx = \int \frac{x+x^2+x^3+x^5}{6x^2+4x^3+3x^4+2x^6}\,dx = \boxed{\frac{1}{12}\ln|6x^2+4x^3+3x^4+2x^6|+C}$$

3.6.7 Nested Radicals

For nested radicals, its sometimes easiest to u-sub the entire integrand and differentiate in a clever way:

$$\int \sqrt{1+\sqrt{1+\sqrt{x}}}\,dx = \int u(8u^7-24u^5+16u^3)du$$

$$\begin{aligned} u &= \sqrt{1+\sqrt{1+\sqrt{x}}} \\ u^2-1 &= \sqrt{1+\sqrt{x}} \\ u^4-2u^2 &= \sqrt{x} \\ u^8-4u^6+4u^4 &= x \\ (8u^7-24u^5+16u^3)du &= dx \end{aligned}$$

For infinitely nested radicals, the function can be rearranged into a more closed form to integrate:

$$f(x) = \sqrt{ax+b\sqrt{ax+b\sqrt{ax+\dots}}} \Longrightarrow f(x) = \sqrt{ax+bf(x)},\text{ solve for } f(x) \Longrightarrow f(x) = \frac{b+\sqrt{b^2+4ax}}{2}$$

$$\int \sqrt{ax+b\sqrt{ax+b\sqrt{ax+\dots}}}\,dx = \int \frac{b+\sqrt{b^2+4ax}}{2}\,dx = \boxed{\frac{b}{2}x+\frac{1}{12a}(b^2+4ax)^{\frac{3}{2}}+C}$$

3.6.8 Undoing a Product Rule

Some integrals look rather imposing but are just the result of using the product rule. Solving these then involve recognizing the product rule and "undoing" it:

$$\int \left(\frac{\sin x}{x}+\cos x\ln x\right)\,dx = \sin x\ln x + C$$

Specifically for e^x we have that:

$$\int (f(x)+f'(x))e^x\,dx = e^x f(x) + C$$

3.6.9 Integration by Parts

Integration by parts is essentially a "reverse product rule". Let $u(x)$ and $v(x)$ be differentiable functions. Then,

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

The Tabular Method is often used when keeping track of multiple integration by parts. Create a table of alternating $+/-$ signs in the first column, successive derivatives of $f(x)$ in the middle column, and successive antiderivatives of $g(x)$ in the last column (ignoring any constants of integration).

$+/-$	D	I
$+$	$f(x)$	$g(x)$
$-$	$f'(x)$	$g^{(-1)}(x)$
$+$	$f''(x)$	$g^{(-2)}(x)$
$-$	$f'''(x)$	$g^{(-3)}(x)$

This can be extended in the obvious way for any number of rows. Now read off from the table in the shapes defined by the distinct colors, and end with the last row as another integral:

$$\int f(x)g(x)\,dx = f(x)g^{(-1)}(x) - f'(x)g^{(-2)}(x) + f''(x)g^{(-3)}(x) - \int f'''(x)g^{(-3)}(x)\,dx$$

This is particularly useful when $f(x) = x^n$ for some integer n, so that $f^{(n+1)}(x) = 0$ and the process eventually terminates.

3.6.10 Trig Function Powers

$$\int \sin^m(x)\cos^n(x)\,dx$$

1. If m or n is odd, repeatedly use the identity $\sin^2(x) = 1 - \cos^2(x)$ or $\cos^2(x) = 1 - \sin^2(x)$ until a singular $\sin(x)$ or $\cos(x)$ remains, then apply the u-substitution $u = \cos(x)$ or $u = \sin(x)$ respectively.
2. If both m and n are even, use the identities $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ and $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$ to reduce the integral into a simpler form.

$$\int \sec^m(x)\tan^n(x)\,dx$$

1. If n is odd, use the identity $\tan^2(x) = \sec^2(x) - 1$ until a singular tangent remains, then apply the u-substitution $u = \sec(x)$.
2. If m is even, use the identity $\sec^2(x) = 1 + \tan^2(x)$ until 2 secants remain, then apply the u-substitution $u = \tan(x)$
3. If m is odd and n is even, use the identity $\tan^2(x) = \sec^2(x) - 1$ until no tangents remain, after which secant reduction is applicable.

Secant reduction is a combination of integration of parts and the identity $\tan^2(x) = \sec^2(x) - 1$, giving the result that for $n \geq 2$,

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

The same methods used for $\sec(x)$ and $\tan(x)$ can be applied to integrals involving $\csc(x)$ and $\cot(x)$.

A neat trick, known as Wallis' Formula can be used to evaluate the definite integral of sin or cos powers. For odd n,

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx = \int_0^{\frac{\pi}{2}} \cos^n(x)\,dx = 1 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot (\ldots) \cdot \frac{n-1}{n}$$

and for even n,

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx = \int_0^{\frac{\pi}{2}} \cos^n(x)\,dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot (\ldots) \cdot \frac{n-1}{n}$$

This formula can be extended to include both $\sin(x)$ and $\cos(x)$ as follows (each "…" carries down by 2 until you reach a 2 or 1):

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\cos^m(x)\,dx = \begin{cases} \dfrac{\pi}{2} \cdot \dfrac{((n-1)(n-3)\ldots)\cdot((m-1)(m-3)\ldots)}{((n+m)(n+m-2)\ldots)} & n \text{ and } m \text{ are both even} \\[2ex] \dfrac{((n-1)(n-3)\ldots)\cdot((m-1)(m-3)\ldots)}{((n+m)(n+m-2)\ldots)} & \text{otherwise} \end{cases}$$

3.6.11 Trig Substitutions

Substitutions involving trig functions can be helpful when evaluating integrals.

- $a^2 - x^2 \longrightarrow x = a\sin(\theta)$
- $a^2 + x^2 \longrightarrow x = a\tan(\theta)$
- $x^2 - a^2 \longrightarrow x = a\sec(\theta)$

Similarly, the hyperbolic trig functions can also be used.

- $x^2 + a^2 \longrightarrow x = a\sinh(\theta)$
- $x^2 - a^2 \longrightarrow x = a\cosh(\theta)$

3.6.12 Bounds Tricks

Bounds tricks usually involve clever u-substitutions of functions that are their own inverses. For example, by substituting $u = -x$,

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)(1+e^x)}\,dx = \int_{-\infty}^{\infty} \frac{e^x}{(1+x^2)(1+e^x)}\,dx$$

$$2I = \int_{-\infty}^{\infty} \frac{1+e^x}{(1+x^2)(1+e^x)}\,dx = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \Big[\arctan(x)\Big]_{-\infty}^{\infty} = \pi \Longrightarrow I = \boxed{\frac{\pi}{2}}.$$

Some common bounds tricks are:

- If there is a $(1+a^x)$ in the denominator and bounds from $-b$ to b, $u = -x$ can be used.
- If the integrand is of the form $\frac{f(x-c)}{f(x-c)+f(d-x)}$ and the bounds are from a to b, where $a+b = c+d$, $u = a+b-x$ is often helpful (some examples are $u = \frac{\pi}{2} - x$ with trig functions over $(0, \frac{\pi}{2})$ and $u = 1-x$ over $(0,1)$).
- If there is a (x^2+a^2) in the denominator and bounds from 0 to ∞ or 1 to a (often with $\ln(x)$ or $\arctan(x)$ in the numerator), the substitution $u = \frac{a}{x}$ is often helpful.

3.6.13 Inverse Functions

If $f(x)$ is continuous and increasing on the interval then

$$\int_a^b f^{-1}(x)\,dx = (bf^{-1}(b) - af^{-1}(a)) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)\,dx$$

This formula can be used for functions in which the inverse function is easier to integrate:

$$\int_0^1 \arctan x\,dx = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{\pi}{4} - \Big[\ln(\sec x)\Big]_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}.$$

3.6.14 Series

Sometimes you can rewrite a function as its series expansion integrate. This is because you can swap the order of the integral and the summation sign (there are specific constraints that must be met for this to be possible, but you definitely won't have to worry about them for FAMAT and competition). For example:

$$\int_0^{\infty} \ln(1+e^{-x})\,dx = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{-nx}}{n}\,dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(-1)^{n+1}e^{-nx}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^{\infty} e^{-nx}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{-1}{n}e^{-nx}\right]_0^{\infty} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \boxed{\frac{\pi^2}{12}}.$$

3.6.15 Complex Numbers

Functions involving complex numbers and are able to be integrated over real intervals.

$$\int x^i\,dx = \frac{1}{1+i}x^{1+i} + C \qquad\qquad \int e^{ix}\,dx = \frac{1}{i}e^{ix} + C$$

Euler's identity ($e^{ix} = \cos(x) + i\sin(x)$) which implies that $\cos(x) = \Re(e^{ix})$ and $\sin(x) = \Im(e^{ix})$ can also be used.

$$\int \cos(\ln(x))\,dx = \int \Re\left(e^{i\ln(x)}\right)\,dx = \Re\left(\int x^i\,dx\right) = \Re\left(\frac{1}{1+i}x^{1+i} + C\right)$$
$$= \Re\left(\frac{1-i}{2}(x)(\cos(\ln(x)) + i\sin(\ln(x))) + C\right) = \boxed{\frac{x}{2}(\cos(\ln(x)) + \sin(\ln(x))) + C}.$$

The e^{ix} expressions for $\sin(x)$ and $\cos(x)$ can also be useful as an alternative to product-to-sum formulas. Consider:

$$\int_0^{2\pi} \cos(x)\sin(3x)\sin(4x)\,dx = \int_0^{2\pi} \frac{e^{ix}+e^{-ix}}{2}\,\frac{e^{3ix}-e^{-3ix}}{2i}\,\frac{e^{4ix}-e^{-4ix}}{2i}\,dx.$$

When integrating from 0 to 2π, all nonzero integer powers of e^{ix} will result in 0, so we only need to consider the constant term in the expansion of the product. As a result:

$$\int_0^{2\pi} \cos(x)\sin(3x)\sin(4x)\,dx = \int_0^{2\pi} \frac{e^{ix}e^{3ix}(-e^{-4ix}) + e^{-ix}(-e^{-3ix})e^{4ix}}{-8}\,dx = \int_0^{2\pi} \frac{-2}{-8}\,dx = \boxed{\frac{\pi}{2}}.$$

Though there is no general formula for this constant term, it can be shown that for any such integral from 0 to 2π, if there is an odd number of $\sin(nx)$ terms or the sum of the coefficients of x is odd, then the integral is 0.

3.6.16 Weierstrass Substitution

For integrals involving trig functions in a rational function, the Weierstrass Substitution $t = \tan(\frac{x}{2})$ can be useful. Under this substitution,

$$\sin(x) = \frac{2t}{1+t^2} \qquad \cos(x) = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}dt.$$

For integrals involving hyperbolic trig functions, there exists a similar substitution $t = \tanh(\frac{x}{2})$. Under this substitution,

$$\sinh(x) = \frac{2t}{1-t^2} \qquad \cosh(x) = \frac{1+t^2}{1-t^2} \qquad dx = \frac{2}{1-t^2}dt.$$

3.6.17 Feynman's Technique

Feynman's technique can be used to solve definite integrals by introducing a new variable a and differentiating with respect to a. For example, consider the integral

$$I = \int_0^\infty \frac{\ln(1+3x^2)}{x^2}\,dx.$$

Define the integral as a function of a:

$$I(a) = \int_0^\infty \frac{\ln(1+ax^2)}{x^2}\,dx.$$

Next, differentiate with respect to a by taking the partial derivative inside the integral, and solve the new definite integral:

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\ln(1+ax^2)}{x^2}\right]dx = \int_0^\infty \frac{1}{1+ax^2}\,dx = \left[\frac{1}{\sqrt{a}}\arctan(\sqrt{a}x)\right]_0^\infty = \frac{\pi}{2\sqrt{a}}.$$

Finally, integrate again with respect to a to recover our original function $I(a)$:

$$I(a) = \int I'(a)\,da = \int \frac{\pi}{2\sqrt{a}}\,da = \pi\sqrt{a} + C.$$

We need an initial condition to solve for C, so we use $I(0) = 0$, which gives us a value of $C = 0$. Therefore, our function is $I(a) = \pi\sqrt{a}$, and our desired answer is $I(3) = \boxed{\pi\sqrt{3}}$.

An arbitrary constant, such as the 3 in the example above, is a common indicator of Feynman integrals. Otherwise, defining our function $I(a)$ may be less obvious, but the general idea is to set a to something that makes the integral "nicer" to evaluate once differentiated with respect to a. To solve for C, we evaluate $I(a)$ at a convenient value, usually one that makes something 0 like $I(0)$ for an integral containing $\sin(ax)$ or $I(\infty)$ for one with e^{-ax}.

3.6.18 Gaussian Integral

It is often useful to know that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

This integral can be related to the normal distribution, and for a normal distribution with mean μ and standard deviation σ,

$$\int_{-\infty}^{\infty} N_{(\mu,\sigma)}(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = 1.$$

The empirical rule is also applicable, which allows us to see that

$$\int_{\mu+\sigma}^{\mu-\sigma} N_{(\mu,\sigma)}(x)\,dx \approx 0.68 \qquad \int_{\mu+2\sigma}^{\mu-2\sigma} N_{(\mu,\sigma)}(x)\,dx \approx 0.95 \qquad \int_{\mu+3\sigma}^{\mu-3\sigma} N_{(\mu,\sigma)}(x)\,dx \approx 0.997$$

3.6.19 Gamma Function

The gamma function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt$$

We can see that $\Gamma(1) = 1$ and by using integration by parts,

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}\,dt = \Big[-t^x e^{-t}\Big]_0^{\infty} + \int_0^{\infty} xt^{x-1}e^{-t}\,dt = x\Gamma(x)$$

This implies that $\Gamma(x) = (x-1)!$ for all positive integers x. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ can also be shown through the Gaussian integral. Euler's reflection formula states that $\Gamma(x)\Gamma(1-x) = \pi\csc(\pi x)$, which may be useful in solving advanced integrals.

3.6.20 Beta Function

The beta function $B(x, y)$ is related to the gamma function and is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

This can be related to the gamma function to give a very useful result.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Thus, $B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$ for positive integers x and y. Through the substitutions $t = \sin^2(u)$, $t = \frac{u}{u+1}$, and $t = u^n$, the beta function can also be expressed in the following equivalent forms:

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin(\theta))^{2x-1}(\cos(\theta))^{2y-1}\,d\theta = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = n\int_0^1 t^{nx-1}(1-t^n)^{y-1}\,dt$$

3.6.21 Glasser's Master Theorem

Glasser's master theorem states that if $u = x - a - \sum_{n=1}^{N} \frac{|a_n|}{x - b_n}$ for real numbers a, a_i, and b_i, then

$$PV \int_{-\infty}^{\infty} F(u)\, dx = PV \int_{-\infty}^{\infty} F(x)\, dx$$

PV denotes the Cauchy principle value, a way to assign values to improper integrals which would otherwise be undefined, but can be ignored for the most part in FAMAT. Note $F(u)dx$, not $F(u)du$, meaning that u can be directly substituted for x without accounting for du. The most common case of this is $u = x - \frac{1}{x}$. Due to a series expansion of cotangent, $u = x - \cot(x)$ is also a valid substitution. For example:

$$\int_0^\infty \frac{x^2}{x^4+1}\, dx = \int_0^\infty \frac{1}{x^2 + \frac{1}{x^2}}\, dx = \int_0^\infty \frac{1}{(x - \frac{1}{x})^2 + 2}\, dx = \int_0^\infty \frac{1}{x^2+2}\, dx = \Big[\frac{1}{\sqrt{2}} \arctan(\frac{x}{\sqrt{2}})\Big]_0^\infty = \frac{\pi}{2\sqrt{2}}$$

3.7 Miscellaneous Definite Integrals

These integrals, like in most advanced integral problems, are rarely used but incredibly useful. You should be able to derive some of them using above techniques, and it can be good practice to try a few of them:

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^4 x + \sin^4 x}\, dx = \int_0^\infty \frac{2x^2}{1+x^4}\, dx = \int_0^\infty \frac{2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}$$

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan(x))\, dx = \frac{\pi}{8}\ln 2$$

$$\int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx = \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = -\frac{\pi}{2}\ln 2$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

$$\int_{-\infty}^{\infty} \cos(x^2)\, dx = \int_{-\infty}^{\infty} \sin(x^2)\, dx = \sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty \frac{\ln x}{ax^2+bx+a}\, dx = 0$$

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\, dx = \frac{\pi \ln a}{2a}$$

$$\int_0^1 (1-x^2)^n\, dx = \frac{2^{2n}\cdot (n!)^2}{(2n+1)!}$$

$$\int_0^\infty e^{-ax}\cos(bx)\, dx = \frac{a}{a^2+b^2}$$

$$\int_0^\infty e^{-ax}\sin(bx)\, dx = \frac{b}{a^2+b^2}$$

$$\int_0^1 x^n \ln^k x\, dx = \frac{(-1)^k k!}{(n+1)^{k+1}}$$

$$\int_0^\infty \frac{x^{k-1}}{1+x^n}\, dx = \frac{\pi}{n}\csc\left(\frac{k}{n}\pi\right)$$

$$\int_0^1 \frac{x^a - 1}{\ln x}\, dx = \ln(a+1)$$

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\, dx = \left(\lim_{x\to\infty} f(x) - f(0)\right)\ln\left(\frac{a}{b}\right)$$

$$\int_0^\pi \frac{\sin(nx)}{\sin(x)}\, dx = \begin{cases} \pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx$$

$$f(x) = f(x \pm \pi) \Longrightarrow \int_0^\infty f(x)\frac{\sin x}{x}\, dx = \int_0^{\frac{\pi}{2}} f(x)\, dx$$

$$f(x) \text{ and } b(x) \text{ are even, } \ g(x) \text{ is odd} \Longrightarrow \int_{-a}^{a} \frac{f(x)}{1+b(x)^{g(x)}}\, dx = \int_0^a f(x)\, dx$$

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx = \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\, dx = \frac{b-a}{2}$$

4 Applications

4.1 Area and Volume

Although many of these formulas are only shown as integrals with respect to x, they can also be used to integrate with respect to y.

4.1.1 Area Between Curves

The area between two functions $f(x)$ and $g(x)$ is

$$\int_a^b |f(x) - g(x)|\, dx$$

You may choose to integrate with respect to the x or y depending on the area required and/or which integral is simpler. When dealing with curves that intersect multiple times, make sure to split up the integral to avoid adding positive and negative areas.

4.1.2 Average Value

The average value of a function $f(x)$ over the interval $[a, b]$ is given by

$$y_{avg} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

4.1.3 Center of Mass or Centroid

Given that $A = \displaystyle\int_a^b f(x) - g(x)\, dx$

$$\overline{x} = \frac{1}{A}\int_a^b x\Big(f(x) - g(x)\Big)\, dx \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\Big([f(x)]^2 - [g(x)]^2\Big)\, dx$$

4.1.4 Solids of Revolution

Using the **disk/washer method** (based on the formula $A = \pi[r_{outer}^2 - r_{inner}^2]$), the volume obtained when rotating the area between two functions $f(x)$ and $g(x)$ about the line $y = k$ is

$$\pi\int_a^b \Big((f(x) - k)^2 - (g(x) - k)^2\Big)\, dx$$

Using the **cylinder/shell method** (based on the formula $A = 2\pi rh$), the volume obtained when rotating the area between two functions $f(x)$ and $g(x)$ about the line $x = k$ is

$$2\pi\int_a^b \Big(f(x) - g(x)\Big)|x - k|\, dx$$

4.1.5 Arc Length

$$L = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

4.1.6 Surface Area

The surface area generated when a curve $f(x)$ is rotated around the x-axis is

$$SA = 2\pi\int y\, ds = 2\pi\int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi\int_c^d y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

4.1.7 Pappus's Centroid Theorems

For simple rotations, or rotations about a slanted line, it is sometimes useful to use Pappus's Centroid Theorems for calculating volume and surface area of solids and surfaces of revolution.

The first theorem of Pappus states that given a curve with arc length s that has a geometric centroid a distance r from the axis of revolution, the resulting surface area SA of the surface of revolution is

$$SA = (2\pi r)s$$

The second theorem of Pappus states that given a shape with area A that has a geometric centroid a distance r from the axis of revolution, the resulting volume V of the solid of revolution is

$$V = (2\pi r)A$$

4.1.8 Common Formulas

Area

1. If $p(x)$ is a polynomial of degree at most 3 with roots r_1 and r_2 and leading coefficient a, then the area under $p(x)$ between r_1 and r_2 is

$$|\frac{2}{3}(r_2 - r_1) \cdot p\left(\frac{r_1 + r_2}{2}\right)|$$

<u>Case of quadratic</u>:

$$\frac{1}{6}|a(r_1 - r_2)^3|$$

<u>Case of cubic</u> (where r_3 is the other root):

$$\frac{1}{6}|a(r_1 - r_2)^3 \cdot \left(\frac{r_1 + r_2}{2} - r_3\right)|$$

Note: These can also be used to find the region bounded by two functions $f(x)$ and $g(x)$ given they intersect at $x = r_1, x = r_2$ and $f(x) - g(x)$ is a polynomial of degree at most 3 with leading coefficient a.

2. Area bounded by a parabola with base B and height h:

$$\frac{2}{3}Bh$$

3. Area bounded by an arch of a sine/cosine curve with base B and height h:

$$\frac{2}{\pi}Bh$$

Volume

1. Volume of a paraboloid (a parabola rotated about its axis of symmetry) with radius r and height h:

$$\frac{1}{2}\pi r^2 h$$

2. Volume of a parabolic spandrel ($y = kx^2$) rotated around the x-axis with base B and height h:

$$\frac{1}{5}\pi B h^2$$

3. Volume of a parabola rotated about the x-axis where its an upside down parabola (V_1) or right side up parabola (V_2) with "base" B and height h:

$$V_1 = \frac{8}{15}\pi B h^2, \;\; V_2 = \frac{4}{5}\pi B h^2$$

4. Volume of a solid with square cross-sections perpendicular to the base, where the base is a circle (V_C) with radius R or an ellipse (V_E) with axes $2a$ and $2b$ where b is parallel to the side of the square:

$$V_C = \frac{16}{3}R^3, \;\; V_E = \frac{16}{3}b^2 a$$

For other types of bases that lie above the x-axis, this formula is the same as rotating the figure around the x-axis.

5. (Conversion) Same problem as 4, but multiply by these constants if using equilateral triangle (ET), Semi-circle (SC), or isosceles triangle with hypotenuse on the base (IT) cross-sections instead:

$$ET = \frac{\sqrt{3}}{4}, \;\; SC = \frac{\pi}{8}, \;\; IT = \frac{1}{4}$$

6. Volume of a spherical cap of height h cut from a sphere of radius R:

$$\frac{\pi h^2(3R - h)}{3}$$

7. Volume of a sphere after a cylindrical hole with height h is drilled through its center (Napkin Ring):

$$\frac{\pi h^3}{6}$$

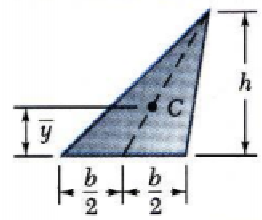
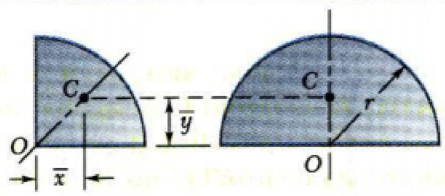
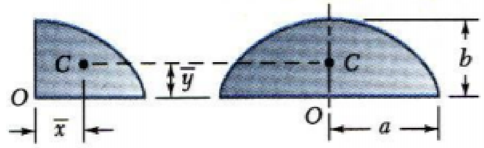
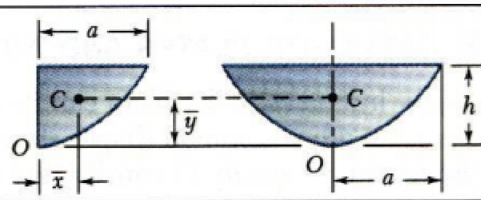
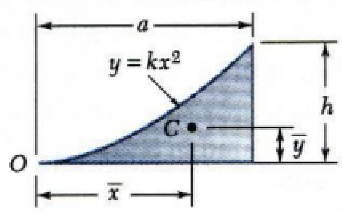
8. Volume V and surface area A of two cylinders of radius r intersecting at right angles (Steinmetz solid or Bicylinder)

$$V = \frac{16}{3}r^3, \quad A = 16r^2$$

9. Volume V and surface area A of three cylinders of radius r intersecting at right angles (Steinmetz solid or Tricylinder)

$$V = 8(2-\sqrt{2})r^3, \quad A = 24(2-\sqrt{2})r^2$$

Centroids

Shape		$\bar{x}$	$\bar{y}$
Triangular area			$\frac{h}{3}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
Semicircular area		0	$\frac{4r}{3\pi}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$
Semielliptical area		0	$\frac{4b}{3\pi}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$
Parabolic area		0	$\frac{3h}{5}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$

1. Centroid of one arc of a sine/cosine curve with height h and base B:

$$\left(\frac{B}{2}, \frac{\pi h}{8}\right)$$

2. Centroid of one half-arc of a sine/cosine curve with height h and base B (B is defined as only half the base before):

$$\left(\frac{2B}{\pi}, \frac{\pi h}{8}\right)$$

3. Centroid of a general spandrel, the area bounded by $y = kx^n$ and the x-axis with base a and height b:

$$\left(\frac{n+1}{n+2}\cdot a, \ \frac{n+1}{2n+1}\cdot\frac{b}{2}\right)$$

4. Centroid of the area bounded by $y = kx^n$ and the y-axis with base a and height b:

$$\left(\frac{n+1}{n+2}\cdot\frac{a}{2}, \ \frac{n+1}{2n+1}\cdot b\right)$$

4.2 Optimization

Optimization involves finding the minimum or maximum value of something given a constraint. There are multiple ways to solve these types of questions.

4.2.1 The Normal Way

This is the most common method of optimization. Given a function $f(x_1, x_2, \ldots, x_n)$, your goal is to express the function in terms of one variable through substitutions of given and derived equations. Once you get $f(x_1)$, you can then find its critical point (where $f'(x_1) = 0$), and use this value of x_1 to solve for the remaining unknowns.

For example, suppose you want to minimize the surface area, S, of a cylinder given a constant volume, V. For height h and radius r, we can write the equations

$$S = 2\pi(rh + r^2) \qquad\qquad V = \pi r^2 h$$

Following the above steps, we get

$$h = \frac{V}{\pi r^2} \Longrightarrow S = 2\pi\left(\frac{V}{\pi r} + r^2\right) \Longrightarrow S' = 2\pi\left(\frac{-V}{\pi r^2} + 2r\right) = 0 \Longrightarrow 2\pi r^3 - V = 0$$

$$r = \left(\frac{V}{2\pi}\right)^{1/3} \Longrightarrow h = \frac{V}{\pi r^2} = \left(\frac{4V}{\pi}\right)^{1/3} \Longrightarrow \boxed{S = \left(54\pi V^2\right)^{1/3}}$$

As with most optimization problems the hardest part is obtaining your equations.

Also, note that if you cannot get the function $f(x_1, x_2 \ldots, x_n)$ in terms of one variable, you can take the partial derivative with respect to each variable (take the derivative normally but treat every other variable as a constant) and then set each equation equal to 0.

4.2.2 Using Implicit Differentiation

This approach is less common, but can be useful when you are more interested in the ratio between two values than the actual final value. As the name suggests, this involves the implicit differentiation of your constraining and given equations with respect to a single variable. After doing so, you will be able to find a relationship between two or more variables. Plugging this into your original equations will give you the values of the variable as well as your optimized answer.

For example, suppose you want to minimize the surface area, S, of a cylinder given a constant volume, V. For height h and radius r, we can write the equations

$$S = 2\pi(rh + r^2) \qquad\qquad V = \pi r^2 h$$

Following the above steps, we get

$$h = \frac{V}{\pi r^2} \Longrightarrow h' = \frac{-2V}{\pi r^3} = \frac{-2h}{r}$$

$$S' = 2\pi(h + rh' + 2r) = 0 \Longrightarrow h - 2h + 2r = 0 \Longrightarrow h = 2r$$

$$V = \pi r^2(2r) \Longrightarrow r = \left(\frac{V}{2\pi}\right)^{1/3} \Longrightarrow S = 2\pi(r(2r) + r^2) = 6\pi r^2 = \boxed{\left(54\pi V^2\right)^{1/3}}$$

Note that we found that $h = 2r$ more easily than the previous method.

4.2.3 Using Lagrange Multipliers

More people use this approach, although it is usually taught in Multivariable Calculus and involves partial derivatives. Once you get the hang of it though, it is definitely viable in a competition setting. Let $f(x_1, x_2, \ldots, x_n)$ be the function you want to optimize and $g(x_1, x_2, \ldots, x_n) = c$ be the given equation. Then, the solution to the optimization problem is the solution to the following equations,

$$\nabla f(x_1, x_2, \ldots, x_n) = \lambda \nabla g(x_1, x_2, \ldots, x_n) \qquad\qquad g(x_1, x_2, \ldots, x_n) = c$$

where λ is an additional variable you have to solve for and ∇ is the gradient operator (basically take the partial derivatives of f and g with respect to each variable). You are never required to solve for the value λ, although sometimes it might be a good intermediate step.

For example, suppose you want to minimize the surface area, S, of a cylinder given a constant volume, V. For height h and radius r, we can write the equations

$$S = 2\pi(rh + r^2) = f(r, h) \qquad\qquad V = \pi r^2 h = g(r, h)$$

Following the above steps, we get

$$\frac{\partial S}{\partial r} = \lambda \frac{\partial V}{\partial r} \implies 2\pi(h + 2r) = 2\pi r h \lambda \implies \lambda = \frac{1}{r} + \frac{2}{h}$$

$$\frac{\partial S}{\partial h} = \lambda \frac{\partial V}{\partial h} \implies 2\pi r = \pi r^2 \lambda \implies \lambda = \frac{2}{r} = \frac{1}{r} + \frac{2}{h} \implies h = 2r$$

$$V = \pi r^2 (2r) \implies r = \left(\frac{V}{2\pi}\right)^{1/3} \implies S = 2\pi(r(2r) + r^2) = 6\pi r^2 = \boxed{\left(54\pi V^2\right)^{1/3}}$$

While this method may seem worse, Lagrange Multipliers is extremely powerful with more than 2 variables or with more than 1 constraint equation. To solve with multiple constraint equations $g_1, g_2, \ldots g_k$ you can just add them on in the following way:

$$\nabla f(x_1, x_2, \ldots, x_n) = \lambda_1 \nabla g_1(x_1, x_2, \ldots, x_n) + \lambda_2 \nabla g_2(x_1, x_2, \ldots, x_n) + \ldots + \lambda_k \nabla g_k(x_1, x_2, \ldots, x_n)$$

4.2.4 Common Formulas

Area and Volume

1. $\star$ Maximum volume V_{max} of a cylinder with radius r and height h inscribed in a sphere of radius R and volume V:

$$V_{max} = \frac{\sqrt{3}}{3} V, \;\; r = \frac{\sqrt{6}}{3} R, \;\; h = \frac{2\sqrt{3}}{3} R$$

2. $\star$ Maximum volume V_{max} of a cone with radius r and height h inscribed in a sphere of radius R and volume V:

$$V_{max} = \frac{8}{27} V, \;\; r = \frac{4}{3} R, \;\; h = \frac{2\sqrt{2}}{3} R$$

3. $\star$ Maximum volume V_{max} of a cylinder with radius r and height h inscribed in a cone of radius R and height H with volume V:

$$V_{max} = \frac{4}{9} V, \;\; r = \frac{2}{3} R, \;\; h = \frac{1}{3} H$$

4. (Conversion) Formulas 1-3 can be converted into the polygon base equivalents, that is, all circular bases are instead n-sided polygons with circumradius r or R, by keeping everything the same except the following conversion after calculating max volumes:

$$\pi \to \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$$

5. $\star$ Given a cylinder with radius r and height h, the dimensions and values for maximum volume V with constant surface area A or minimum surface area with constant volume:

$$V = \left(\frac{1}{54\pi} A^3\right)^{\frac{1}{2}}, \;\; A = \left(54\pi V^2\right)^{\frac{1}{3}}, \;\; \frac{r}{h} = \frac{1}{2}$$

6. ⋆ Given an open top cylinder with radius r and height h, the dimensions and values for maximum volume V with constant surface area A or minimum surface area with constant volume:

$$V = \left(\frac{1}{27\pi}A^3\right)^{\frac{1}{2}}, \quad A = \left(27\pi V^2\right)^{\frac{1}{3}}, \quad \frac{r}{h} = 1$$

7. Given a cone with radius r and height h, the dimensions and values for maximum volume V with constant surface area A or minimum surface area with constant volume:

$$V = \left(\frac{1}{72\pi}A^3\right)^{\frac{1}{2}}, \quad A = \left(72\pi V^2\right)^{\frac{1}{3}}, \quad \frac{r}{h} = \frac{1}{2\sqrt{2}}$$

8. (Conversion) Formulas 5-7 can be converted into the polygon base equivalents, that is, the circular base(s) are instead n-sided polygons with side length s, in the following way:

$$\pi \to n\tan\left(\frac{\pi}{n}\right), \quad \frac{s}{h} = \frac{r}{h}\cdot 2\tan\left(\frac{\pi}{n}\right)$$

9. Minimum volume V_{min} of a cone with radius r and height h circumscribed about a sphere of radius R and volume V:

$$V_{min} = 2V, \quad r = R\sqrt{2}, \quad h = 4R$$

10. Dimensions of the maximum volume cone cut from a circle of radius R, and angle θ cut from the circle to create the cone:

$$r = R\sqrt{\frac{2}{3}}, \quad h = R\sqrt{\frac{1}{3}}, \quad \theta = \left(2 - \sqrt{\frac{8}{3}}\right)\pi$$

11. Maximum area of a rectangle that can be made with p length of fencing along a line that requires no fencing (ie. a river or wall):

$$\frac{p^2}{8}$$

12. Maximum area of a rectangle inscribed in a right triangle with area A with one side on each leg and a vertex on the hypotenuse:

$$\frac{A}{2}$$

13. Maximum area of a rectangle inscribed in any triangle with area A such that two vertices lie on one side and one vertex lies on each of the other two sides:

$$\frac{A}{2}$$

14. Area A and dimensions of the largest rectangle of length l and height h inscribed in an upside-down parabola with base B and height H:

$$A = \frac{2\sqrt{3}}{9}BH, \quad h = \frac{2}{3}H, \quad l = \frac{\sqrt{3}}{3}B$$

15. Area A and dimensions of the largest trapezoid of top base b and height h inscribed in an upside-down parabola with base B and height H:

$$A = \frac{16}{27}BH, \quad h = \frac{8}{9}H, \quad b = \frac{1}{3}B$$

16. Maximum area of a shape inscribed in an ellipse with semi-major and minor axes a and b given that the maximum area of the shape inscribed in a circle of radius r is kr^2:

$$kab$$

17. ⋆ Minimum area A of a triangle bounded by the x and y axes and a line with slope m passing through the point (a, b):

$$A = 2ab, \quad m = -\frac{b}{a}$$

18. ⋆ Maximum area and dimensions of a Norman window (semicircle on top of a rectangle) with semicircle radius r and rectangle height y with set perimeter p:

$$A = \frac{p^2}{2(\pi+4)}, \quad r = y = \frac{p}{\pi+4}$$

19. ⋆ Maximum area and dimensions of a Norman window (semicircle on top of a rectangle) with semicircle radius r and rectangle height y with set perimeter p, except the diameter of the semicircle is included in the set perimeter:

$$A = \frac{p^2}{2(\pi+8)}, \quad 2r = y = \frac{2p}{\pi+8}$$

Other

1. A concert can sell T tickets for D dollars each. When the price increases d dollars, the number of tickets sold drop by t. The price of each ticket P to maximize revenue R is:

$$P = D + \frac{d}{2}\left(\frac{T}{t} - \frac{D}{d}\right), \quad R = \frac{1}{4dt}(Dt + Td)^2$$

2. (Utility Function in Economics) Given the budget constraint $p_x x + p_y y = B$, if you want to maximize the utility function $U(x, y)$, the following holds:

$$\frac{\partial U/\partial x}{\partial U/\partial y} = \frac{p_x}{p_y}$$

Specifically, given a *Cobb-Douglas* utility function $U(x, y) = kx^a y^b$, you have:

$$\frac{ay}{bx} = \frac{p_x}{p_y}$$

3. Let (x_0, y_0) be a point on the graph $f(x) = ax^2 + bx + c$. Then the value of x_0 such that the line tangent to $f(x)$ at x_0 goes through (x_1, y_1) is:

$$x_0 = x_1 \pm \sqrt{\frac{f(x_1) - y_1}{a}}$$

If a triangle is then formed with (x_1, y_1) and the two points of tangency, the area of the triangle is:

$$\frac{2}{\sqrt{|a|}}(|f(x_1) - y_1|)^{\frac{3}{2}}$$

4. Let $P = (x_1, y_1)$ be a point on the graph $y = kx^2$ and points $Q = (a, b)$ and $R = (c, d)$ be such that $a < c$ and the line QR never intersects the graph $y = kx^2$. Then the value of x_1 that minimizes the area of ΔPQR is:

$$x_1 = \frac{1}{2k} \cdot \left(\frac{d - b}{c - a}\right) = \frac{1}{2k} \cdot \text{(slope from Q to R)}$$

5. Let (x_1, y_1) be a point on the graph $y = a\sqrt{x}$ and $(x_0, 0)$ be a point on the $x-axis$. Then the value of x_1 that minimizes the distance D between (x_1, y_1) and $(x_0, 0)$ is:

$$x_1 = x_0 - \frac{a^2}{2}, \quad D = a\sqrt{x_0 - \frac{a^2}{4}}$$

6. Given a $a \times b$ sheet of paper with squares of side length s cut from the corners, value of s that maximizes the volume of the open-topped box formed by folding the flaps up:

$$s = \frac{a + b - \sqrt{a^2 - ab + b^2}}{6}$$

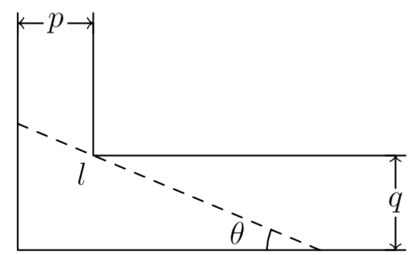
When $a = b$, $s = a/6$

7. Given a paper with x margins on top and bottom and y margins on the sides that must have a print area A, the length L (vertical) and width W (horizontal) that minimizes the paper used are:

$$L = 2x + \sqrt{A \cdot \frac{x}{y}}, \quad W = 2y + \sqrt{A \cdot \frac{y}{x}}$$

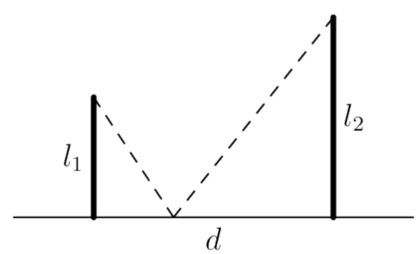
8. ⋆ Length l of the longest ladder that can pass through the junction of two hallways of widths p and q, and its angle of inclination θ:

$$l = (p^{2/3} + q^{2/3})^{3/2}, \quad \tan^3\theta = \frac{p}{q}$$



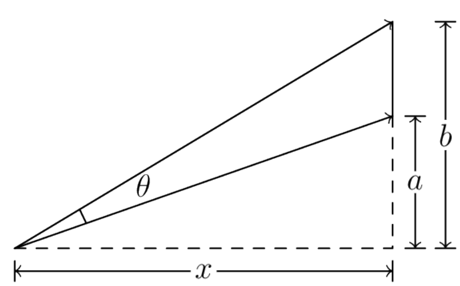
9. Given two poles of length l_1 and l_2 a distance d apart, minimum length of wire that runs from the top of one pole to the ground and back up to the top of the other pole:

$$\sqrt{(l_1 + l_2)^2 + d^2}$$



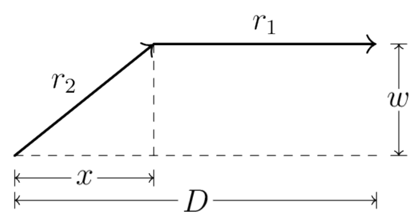
10. ⋆ The distance x that you must stand in front of a picture b units tall from the top and a units tall from the bottom to maximize the viewing angle θ shown below:

$$x = \sqrt{ab}$$



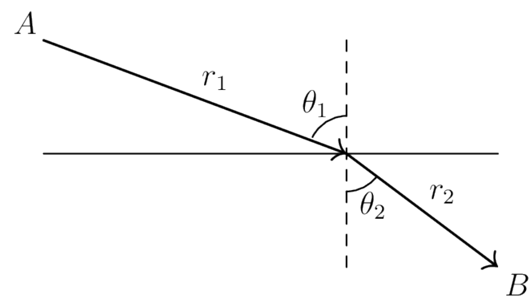
11. ⋆ Given you can cut across in a medium at a speed r_2 (ie. in the water) and are a width w from a place where you can travel at a faster speed r_1 (ie. the shoreline), the horizontal distance x traveled when cutting across to get to a point D distance horizontally in the minimum amount of time T:

$$x = \frac{r_2 w}{\sqrt{r_1^2 - r_2^2}}, \quad T = \frac{w}{r_1 r_2}\sqrt{r_1^2 - r_2^2} + \frac{D}{r_1}$$



12. (Snell's Law) Given two different mediums in which you travel at speeds r_1 and r_2 and must take the minimum amount time to pass through them from point A to point B, the relationship between the angles θ_1 and θ_2 shown below is:

$$\frac{\sin\theta_1}{r_1} = \frac{\sin\theta_2}{r_2}$$



4.3 Related Rates

Related rates problems involve finding the rate of something given the values and rates of something else. Solving these usually involve finding a constraint, such as triangle ratios, and differentiating to solve for the rate. In other words, if you have a two variables x and y that vary with time t, and a constraint $f(x) = g(y)$, you can solve for the rate of one in terms of the other using $f'(x)x' = g'(y)y'$

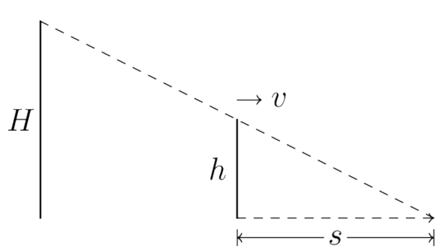
4.3.1 Common Formulas

1. Rate of change of the height/depth of water in a container given the rate of change of the volume V' and area of the surface of the water A:

$$h' = \frac{V'}{A}$$

2. A man of height h is walking away from a light source with height H at a rate v. The rate of change of his shadow s' is:

$$s' = \frac{hv}{H-h}$$

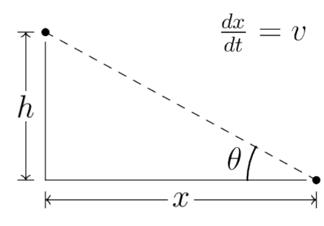


3. Given two objects that start at the same point and move away from each other at constant rates a' and b' where the angle between them is θ, the rate of change of the distance D between them is:

$$D' = \sqrt{(a')^2 + (b')^2 - 2a'b'\cos(\theta)}$$

4. ⋆ Given an object at a fixed height h, an observer on the ground x distance horizontally from the object, and that the distance x is changing at a rate v, the rate of change of the distance d between the object and observer and the angle of inclination θ ($v > 0$ means that x is growing):

$$d' = \frac{xv}{\sqrt{h^2 + x^2}} = \frac{xv}{d}, \quad \theta' = -\frac{hv}{h^2 + x^2} = -\frac{hv}{d^2}$$



5. A lighthouse a distance D from the shore is rotating its light at a rate θ'. The rate x' at which its light moves across the shore when it is x distance along the shore is:

$$x' = \left(\frac{D^2 + x^2}{D}\right)\theta'$$

4.4 Probability and Expected Value

A *probability density function* (PDF) is the function $f(x)$ that gives the range of probabilities for a continuous random variable x. In order to be valid, $f(x)$ must always be non-negative, and $\int_{-\infty}^{\infty} f(x)\,dx = 1$, meaning the probability of the entire sample space is 1. However, the function need not be continuous, as seen in the Dirac Delta function. The probability that x is between two values a and b is simply:

$$\int_a^b f(x)\,dx$$

A *cumulative distribution function* (CDF) is defined as $CDF(x) = \int_{-\infty}^{x} PDF(t)\,dt$.

The *mean* (μ) or *expected value* ($E[x]$) of a PDF function $f(x)$ is mathematically the same as the x coordinate of the centroid ($\overline{x}$). In other words,

$$\mu_x = E[x] = \overline{x} = \int_{-\infty}^{\infty} xf(x)\,dx$$

The *mode* of a PDF function is simply the x coordinate that corresponds to the maximum value of that function. The *median* of a PDF function is the x coordinate where $CDF(x) = 0.5$ (which is different from the "balancing point", or the mean). The *variance* (σ^2) of a PDF simply measures its spread and follows the formula

$$Var(x) = E[x^2] - E[x]^2$$

4.4.1 Computing Probabilities

When computing probabilities or expected values using integration, we must first define our random variable, then define the PDF such that its integral over the domain is equal to 1, and finally carry out the calculation. For example, consider a point randomly chosen within a sphere of radius 1. Find the expected value of the distance from the point to the edge of the sphere, and the probability that this distance is less than $\frac{1}{2}$.

We define our random variable r on the domain $(0, 1)$. Since all points within the sphere at a radius of r exist on a spherical shell with area $4\pi r^2$, the probability of picking a point with radius r is proportional to r^2. Thus, we determine our PDF:

$$f(r) = kr^2 \Longrightarrow \int_0^1 kr^2\,dr = 1 \Longrightarrow k = 3,\ f(r) = 3r^2$$

The distance from a point at radius r to the edge of the sphere is $1 - r$. Additionally, this distance is less than $\frac{1}{2}$ when $r > \frac{1}{2}$. Using this, we can compute the desired expected value and probability:

$$\int_0^1 (1-r)f(r)\,dr = \int_0^1 (1-r)3r^2\,dr = \boxed{\frac{1}{4}}$$

$$\int_{\frac{1}{2}}^1 f(r)\,dr = \int_{\frac{1}{2}}^1 3r^2\,dr = \boxed{\frac{7}{8}}$$

Note that we did not compute the expected value of the random variable r itself, as described in the previous section, but rather a function of the random variable, in this case $1 - r$. This applies in general; if we wanted to compute the expected value of the volume of a sphere internally tangent to the given sphere with a center at our random point, we would compute the integral $\int_0^1 \frac{4}{3}\pi(1-r)^3 f(r)\,dr$. Also note that as a shortcut, our PDF $f(r) = 3r^2$ is equal to $\frac{d(r^3)}{dr}$ since $d(r^3)$ represents a spherical shell of radius r and thickness dr, and $r^3\big|_0^1 = 1$.

4.4.2 Exponetial Distribution

The PDF of an exponential distribution is defined as

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

for some positive real number λ.
The following hold true:

1. The mean is $\frac{1}{\lambda}$.
2. The mode is 0.
3. The median is $\frac{\ln 2}{\lambda}$.
4. The variance is $\frac{1}{\lambda^2}$.

4.4.3 Beta Distribution

The PDF of a Beta distribution can take the form

$$f(x) = \begin{cases} Kx^n(1-x)^m & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

for positive integers n and m. In order to be valid, $K = \frac{(n+m+1)!}{n! \cdot m!}$.
The following hold true:

1. The mean is $\frac{n+1}{n+m+2}$.
2. The mode is $\frac{n}{n+m}$.
3. The variance is $\frac{(n+1)(m+1)}{(n+m+2)^2(n+m+3)}$.

4.5 Physics

4.5.1 Kinematics

Position $x(t)$, velocity $v(t)$, and acceleration $a(t)$ are related by differentiation:

$$x'(t) = v(t), \quad v'(t) = a(t)$$

Speed is defined as $|v(t)|$. Therefore, speed increases only when $v(t)$ and $a(t)$ are the same sign and decreases only when they are different signs.

Displacement s and total distance d of an object over a time interval $[t_1, t_2]$ can then be calculated by:

$$s = \int_{t_1}^{t_2} v(t)\,dt, \qquad d = \int_{t_1}^{t_2} |v(t)|\,dt$$

In practice, the second integral is found by finding the roots of $v(t)$ and splitting the integral.

Often times you will see problems dealing with a constant acceleration a. The formulas for this type of motion are

$$v(t) = v_0 + at, \quad x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

Where x_0 and v_0 are initial displacements. An example of this is projectile motion. Consider an object in parabolic motion after being thrown. Let v_0 be its initial speed, θ its launch angle, y_0 its initial height, and g be gravity (usually given as $-10\,\mathrm{m/s^2}$, $-9.8\,\mathrm{m/s^2}$ or $-32\,\mathrm{ft/s^2}$). Then the functions for the vertical $y(t)$ and horizontal $x(t)$ displacement are (Note that $v_0 \cos\theta$ and $v_0 \sin\theta$ are just the initial horizontal and vertical speeds respectively and these may already be given):

$$x(t) = x_0 + (v\cos\theta)t, \quad y(t) = \frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0$$

In particular, if T is the total time in air, H the maximum height, and R the range (maximum horizontal distance), then:

$$T = \frac{2v\sin\theta}{g}, \quad H = \frac{v^2 \sin^2\theta}{2g}, \quad R = \frac{v^2 \sin 2\theta}{g}$$

4.5.2 Mass Density

The linear mass density λ of an object is essentially the mass cross-sectional area and, if constant throughout, is given by $\frac{M}{L}$, where M is the total mass and L is the length. The linear mass density, however, can change through the length of the object, and can be modeled by $\lambda(x)$. The total mass of an object M is given by:

$$M = \int_0^L \lambda(x)\,dx$$

4.5.3 Work and Energy

Work, in layman terms, is the amount of energy needed to do something. If $F(x)$ is the force that is being applied at a distance x over a total distance of L then

$$W = \int_0^L F(x)\,dx$$

A common problem that uses this involves lifting something up using a rope. Given a weight with weight W and a rope with length L and linear mass density λ, the amount of work to lift the weight the total length of rope is:

$$WL + \frac{\lambda L^2}{2}$$

Sometimes this problem includes an object with weight W_0 on the bottom and weight W_1 on the top, where $W_0 > W_1$. For example, say a bucket that is leaking water as it reaches the top. Then the total amount of work to lift the weight the total length of the rope is:

$$\left(\frac{W_0 - W_1}{2}\right) L + \frac{\lambda L^2}{2}$$

The total amount of energy E_T of something is equal to the sum of its kinetic energies K and potential energies U. If $U(t)$ is the potential energy of an object at a distance x, then

$$U'(x) = -F(x)$$

where $F(x)$ is the force applied on the object at a distance x.

In particular, for a spring that is stretched a distance x, $F(x) = -kx$ and $U(x) = \frac{1}{2}kx^2$, where k is the spring constant of the spring.

5 Polar and Parametric

5.1 Polar

5.1.1 Derivatives

Using the substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

5.1.2 Area Between Curves

The area of a polar curve is

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

The area between two polar functions is

$$A = \int_\alpha^\beta \frac{1}{2}(r_0^2 - r_1^2) \, d\theta$$

5.1.3 Center of Mass

$$\overline{x} = \frac{\frac{1}{3}\displaystyle\int_\alpha^\beta r^3 \cos(\theta) \, d\theta}{\frac{1}{2}\displaystyle\int_\alpha^\beta r^2 \, d\theta} \qquad \overline{y} = \frac{\frac{1}{3}\displaystyle\int_\alpha^\beta r^3 \sin(\theta) \, d\theta}{\frac{1}{2}\displaystyle\int_\alpha^\beta r^2 \, d\theta}$$

5.1.4 Arc Length

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

5.1.5 Surface Area

If the curve is revolved around the x-axis,

$$SA = 2\pi \int_\alpha^\beta r\sin(\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

If the curve is revolved around the y-axis,

$$SA = 2\pi \int_\alpha^\beta r\cos(\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

5.1.6 Common Formulas from Polar Graphs

1. Area of a limacon ($r = a + b\sin(\theta)$, $\left|\frac{a}{b}\right| \geq 1$):

$$\left(a^2 + \frac{b^2}{2}\right)\pi$$

2. Areas of a limacon ($r = a + b\sin(\theta)$, $\left|\frac{a}{b}\right| < 1$):

Inner Loop

$$\left(a^2 + \frac{b^2}{2}\right)\cos^{-1}\left(\frac{a}{b}\right) - \frac{3}{2}a\sqrt{b^2 - a^2}$$

Outer Loop (Total Area)

$$\left(a^2 + \frac{b^2}{2}\right)\left(\pi - \cos^{-1}\left(\frac{a}{b}\right)\right) + \frac{3}{2}a\sqrt{b^2 - a^2}$$

In Between (Outer − Inner)

$$\left(a^2 + \frac{b^2}{2}\right)\left(\pi - 2\cos^{-1}\left(\frac{a}{b}\right)\right) + 3a\sqrt{b^2 - a^2}$$

3. Length of a cardioid ($r = a + a\cos(\theta)$):

$$8a$$

4. Area of one petal of a polar rose curve ($r = a\sin(n\theta)$):

$$\frac{\pi a^2}{4n}$$

5. Area of a lemniscate ($r^2 = a^2\sin(2\theta)$):

$$a^2$$

6. Area between two cardioids ($a + a\cos(\theta)$ and $a - a\cos(\theta)$):

$$a^2\left(\frac{3}{2}\pi - 4\right)$$

7. Area between two cardioids ($a + a\cos(\theta)$ and $a + a\sin(\theta)$):

$$a^2\left(\frac{3}{2}\pi - 2\sqrt{2}\right)$$

Note: swapping + and − or sin *and* cos *in the polar equations does not change areas or lengths.*

5.2 Parametric

5.2.1 Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

5.2.2 Area Between Curves

The signed area under a parametric curve from $t = a$ to $t = b$ is:

$$A = \int_a^b y\,dx = \int_a^b y(t)x'(t)\,dt$$

Keep in mind areas can overlap if the curve crosses over itself over time, and are negative when $x'(t)$ is negative.

5.2.3 Center of Mass

$$\overline{x} = \frac{1}{A}\int_a^b x(t)y(t)x'(t)\,dt \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}(y(t))^2x'(t)\,dt$$

5.2.4 Solids of Revolution

If the curve is revolved around the x-axis,

$$V = \pi\int_a^b (y(t))^2x'(t)\,dt$$

If the curve is revolved around the y-axis,

$$V = \pi\int_a^b (x(t))^2y'(t)\,dt$$

5.2.5 Arc Length

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

5.2.6 Surface Area

If the curve is revolved around the x-axis,

$$SA = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

If the curve is revolved around the y-axis,

$$SA = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

5.2.7 Cycloids!

A *cycloid* is the result of rolling a circle on the x-axis and tracing a point on the circle.

Area A and length L of one arc of a cycloid created by rotating a circle of radius r (has parameterization $x = r(t - \sin t)$, and $y = r(1 - \cos t)$):

$$A = 3\pi r^2, \;\; L = 8r$$

The centroid of one arc of a cylcoid C_1 and of one half arc of a cycloid C_2 are:

$$C_1 = \left(\pi r,\; \frac{5}{6}r\right), \;\; C_2 = \left(\left(\frac{\pi}{2} + \frac{8}{9\pi}\right)r,\; \frac{5}{6}r\right)$$

Interestingly, Huygens found that the cycloid is the solution to the Tautochrone problem (A curve for which the time it takes for an object to reach the bottom by gravity is independent of the starting position) in 1659, and Bernoulli found that it is also the solution to the Brachistochrone problem (A curve that takes the least time to get to the bottom by gravity) in 1697. As a consequence of being the Brachistochrone curve, it follows Snell's Law everywhere.

There are several variations on the cycloid:

1. An *epicycloid* is the result of rotating a circle of radius r *around* another circle of radius R. If $R = kr$ and k is an integer, the epicycloid will have k cusps, and its area A and length L are:

$$A = (k+1)(k+2)\pi r^2, \;\; L = 8(k+1)r$$

2. A *hypocycloid* is the result of rotating a circle of radius r *inside* another circle of radius R. If $R = kr$ and k is an integer, the hypocycloid will have k cusps, and its area A and length L are:

$$A = (k-1)(k-2)\pi r^2, \;\; L = 8(k-1)r$$

3. An *astroid* (usually given as $x^{2/3} + y^{2/3} = a^{2/3}$, or its parameterization $x(t) = a\cos^3(t)$, $y(t) = a\sin^3(t)$) is a hypocycloid where $k = 4$, and its area A and length L are:

$$A = \frac{3}{8}\pi a^2, \;\; L = 6a$$

6 Sequences and Series

6.1 Taylor Series

The Taylor Series of a function $f(x)$ centered at c is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

The Maclaurin Series is the special case when $c = 0$.

6.1.1 Common Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad x \in \mathbb{R}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad x \in (-1,1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad x \in (-1,1]$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad x \in [-1,1]$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \qquad x \in \mathbb{R}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \qquad x \in \mathbb{R}$$

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = 1 + 2x + 6x^2 + 20x^3 + \dots \qquad x \in \left[-\frac{1}{4}, \frac{1}{4}\right)$$

$$\arcsin x = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n+1}}{4^n(2n+1)} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \dots \qquad x \in [-1,1]$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

6.1.2 Deriving Taylor Series

The Taylor Series for most functions can be easily derived from one of the basic Taylor Series above. This can be done by either multiplying by or "plugging in" factors of x^n. For example:

$$xe^{-x^2} = x\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!}$$

The same process can be done in reverse in order to rewrite some infinite sums as the Taylor Series of a function.

6.1.3 Using Taylor to Compute Higher Order Derivatives

The value of $f^{(n)}(c)$ will always be the coefficient of the x^n term in the Taylor Series for $f(x)$ centered at c times $n!$. For example:

$$\frac{d^{2019}}{dx^{2019}}\left[x^3\cos x^2\right]_{x=0} \longrightarrow \text{rewrite as Taylor Series} \longrightarrow \sum_{x=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n)!} \longrightarrow \text{find coeff of } x^{2019} \text{ term}$$

$$\longrightarrow \frac{1}{1008!} \longrightarrow \text{multiply by 2019!} \longrightarrow \boxed{\frac{2019!}{1008!}}$$

6.2 Series Tests

6.2.1 n-th Term Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ will diverge. This test cannot prove convergence, only divergence.

6.2.2 Integral Test

If $f(x)$ is continuous, positive, and decreasing on $[k, \infty)$, then $\displaystyle\int_k^{\infty} f(x)\,dx$ and $\sum_{n=k}^{\infty} f(n)$ either both converge or both diverge.

6.2.3 p-series Test

A p-series has the form $\sum_{n=k}^{\infty}\frac{1}{n^p}$. If $p > 1$, the series converges. If $p \leq 1$, the series diverges.

6.2.4 Direct Comparison Test

For two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n,

1. If $\sum b_n$ converges then $\sum a_n$ converges
2. If $\sum a_n$ diverges then $\sum b_n$ diverges

6.2.5 Limit Comparison Test

For two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n, if $\lim_{n\to\infty}\frac{a_n}{b_n}$ is positive and finite, then both series converge or both series diverge.

6.2.6 Alternating Series Test

For a series $\sum a_n$ such that $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ and $b_n \geq 0$ for all n, if b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum a_n$ converges.

If $\sum |a_n|$ converges then $\sum a_n$ converges and $\sum a_n$ is absolutely convergent. If $\sum a_n$ converges but $\sum |a_n|$ diverges, $\sum a_n$ is conditionally convergent.

6.2.7 Ratio Test

Define $L = \lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$. If $L < 1$ the series is absolutely convergent, if $L > 1$ the series is divergent, and if $L = 1$ the test is inconclusive.

6.2.8 Root Test

Define $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. If $L < 1$ the series is absolutely convergent, if $L > 1$ the series is divergent, and if $L = 1$ the test is inconclusive.

6.2.9 Kummer's Test (haha)

Given a series with positive terms u_n and a positive sequence $\{a_n\}$, then

$$\lim_{n\to\infty} \left| a_n \frac{u_n}{u_{n+1}} - a_{n+1} \right| \longrightarrow \begin{cases} > 0, & \text{series converges} \\ < 0, & \text{and } \sum 1/a_n \text{ diverges, } \sum u_n \text{ diverges} \\ = 0, & \text{inconclusive} \end{cases}$$

6.2.10 Raabe's Test

This is a special case of Kummer's where $a_n = n$:

$$\lim_{n\to\infty} \left| n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right| \longrightarrow \begin{cases} > 1, & \text{converges} \\ < 1, & \text{diverges} \\ = 1, & \text{inconclusive} \end{cases}$$

6.2.11 Cauchy - D'Alembert Theorem

For a sequence a_n, if the ratio and root test both yield a limit that exists, then the limits are equivalent:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

6.2.12 Stolz - Cesàro Theorem

To be used in conjunction with ratio test: Assume the sequence $\{b_n\}$ is strictly monotone and that either $\{b_n\}$ is divergent OR that $\{a_n\}$ and $\{b_n\}$ both converge to 0. If the following limit is finite, then:

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{a_n}{b_n}$$

6.3 Interval and Radius of Convergence

The interval of convergence for a series $u(x)$ is the interval of x values in which $u(x)$ converges. The radius of convergence is half the length of the interval. These intervals are most often found using either a known Taylor Series or the ratio test:

$$\sum_{n=1}^{\infty} \left(\frac{(1-x)^{\frac{n}{3}}}{n3^{n-1}} \right) \longrightarrow \lim_{n\to\infty} \left| \frac{\dfrac{(1-x)^{\frac{n+1}{3}}}{(n+1)3^n}}{\dfrac{(1-x)^{\frac{n}{3}}}{n3^{n-1}}} \right| = \lim_{n\to\infty} \left| \frac{(1-x)^{\frac{1}{3}}}{3} \right| < 1 \longrightarrow -26 < x < 28$$

The radius of convergence is unaffected by the bounds, but when providing the interval, you must plug back in the bounds to check if it converges at them or not. In this case, the series does converge at $x = 28$ by the alternating series test, and diverges at $x = -26$ by the harmonic series. Thus the interval of convergence is $\boxed{(-26, 28]}$.

6.3.1 Literally the Radius of Convergence

One rare problem on radius of convergence that cannot easily be solved with a series test involves a circle in the Argand plane. Consider finding the radius of convergence of the Taylor Series for $1/(x^2+9)$ centered at $x = 4$. The trick with this is to consider complex numbers. The imaginary number for which this can be seen to diverge is $3i$ since that would divide by zero. We can then draw a circle in the Argand plane centered at 4 with $3i$ on its border, making it the largest circle centered at 4 such that every complex number inside it converges. We can then find the *literal radius* of convergence by finding the distance between 4 and $3i$ in the plane, giving $\boxed{5}$. In fact, this method is the actual definition of radius of convergence.

6.4 Lagrange Error Bound

The maximum error of an nth degree Taylor approximation centered at a in approximating $f(x)$ is equal to

$$\left|\frac{\max(f^{n+1}(z))(x-a)^{n+1}}{(n+1)!}\right|$$

where $\max(f^{n+1}(z))$ is the maximum value of the $n{+}1^{th}$ derivative of f between a and x.

6.5 Special Types of Series

6.5.1 Faulhaber's Formula

For any integer power p, we can express the sum of the first n p^{th} powers as a polynomial in n, with

$$1^p + 2^p + \cdots + n^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + O(n^{p-1})$$

where the $O(n^{p-1})$ denotes that all other terms in the polynomial have degree at most $p-1$. The above formula is most useful when calculating limits; for an example, consider

$$\lim_{n\to\infty} \frac{(1^p + 2^p + \cdots + n^p) - \frac{n^{p+1}}{p+1}}{n^p},$$

which Faulhaber's shows is $\frac{1}{2}$. This formula also holds for non-integer powers p, although in that case we no longer have a polynomial in n, but rather an infinite series of terms with degree $p+1, p, p-1, \ldots$. However, the asymptotics shown above (the coefficients of n^{p+1} and n^p) still hold.

6.5.2 Power Geometric

$$\sum_{n=0}^{\infty} n^k r^n = \frac{r \cdot P_k(r)}{(1-r)^{k+1}}$$

where $P_k(r)$ is the k^{th} *Eulerian polynomial.* In particular,

$$\sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}$$

$$\sum_{n=0}^{\infty} n^2 r^n = \frac{r^2 + r}{(1-r)^3}$$

$$\sum_{n=0}^{\infty} n^3 r^n = \frac{r^3 + 4r^2 + r}{(1-r)^4}$$

$$\sum_{n=0}^{\infty} n^4 r^n = \frac{r^4 + 11r^3 + 11r^2 + r}{(1-r)^5}$$

Also, $P_k(1) = k!$.

6.5.3 $n^k/n!$

$$\sum_{n=0}^{\infty} \frac{n^k}{n!} = (B_k)e$$

where B_k is the k^{th} *Bell number.* In particular, $B_0 = B_1 = 1$, $B_2 = 2$,$B_3 = 5$, $B_4 = 15$, and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

6.5.4 sinc(x)

The function sinc(x) is defined as:

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

The following properties hold:

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) = \prod_{n=1}^{\infty}\cos\left(\frac{x}{2^n}\right)$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \sum_{n=-\infty}^{\infty}\frac{\sin n}{n} = \sum_{n=-\infty}^{\infty}\left(\frac{\sin n}{n}\right)^2 = \pi$$

$$\sum_{n=1}^{\infty}\frac{\sin n}{n} = \sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\right)^2 = \frac{\pi - 1}{2}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\sin n}{n}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\sin n}{n}\right)^2 = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\sin n}{n}\right)^3 = \frac{1}{2}$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^3 dx = \frac{3}{4}\pi, \quad \int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^4 dx = \frac{2}{3}\pi$$

6.5.5 Other Trigonometric Products and Series

$$\sin x = x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) \qquad \cos x = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(n-\frac{1}{2})^2\pi^2}\right)$$

$$\cot x = \sum_{n=-\infty}^{\infty}\frac{1}{x + n\pi} \qquad \tan x = \sum_{n=-\infty}^{\infty}\frac{-1}{x + (n+\frac{1}{2})\pi}$$

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right) = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots$$

6.5.6 $1/n^2$ and its Forms

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \qquad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{24} \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

6.6 Riemann Rearrangement Theorem

If a series $\sum a_n$ is conditionally convergent, then the terms of the sequence a_n can be rearranged to form any real number.

7 Differential Equations

7.1 Euler's Method

Euler's Method can be used to approximate values of a function given a function $f'(x,y) = \frac{dy}{dx}$, an initial point (x_0, y_0), and a step size h. The recursion is given as follows:

$$f(x+h) \approx f(x) + h \cdot f'(x,y)$$

7.2 Logistic Growth

A logistic curve is a S-shaped curve that follows the following equations

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right) \qquad P(t) = \frac{A}{1 + (\frac{A-P_0}{P_0})e^{-kt}}$$

where P represents the population, t represents time, A represents the carrying capacity ($\lim_{t\to\infty} P(t)$), P_0 is the initial population, and k is a constant. The function has an inflection point at $P(t) = \frac{A}{2}$, which represents the point of fastest growth.

7.3 Newton's Law of Cooling

When an object is placed in an environment with a different temperature, the following equations are applicable:

$$\frac{dT}{dt} = k(T - T_S) \qquad T(t) = T_S + (T_0 - T_S)e^{kt}$$

where T represents the temperature of the object, t represents time, T_S represents the temperature of the surroundings, T_0 represents the initial temperature of the object, and k is a constant.

7.4 Tanks and Mixtures

Given a situation where a tank's volume is initially V_0 and modeled by the function $V(t)$, let the amount of solute initially be X_0 and modeled by $X(t)$. Additionally, liquid comes in at a rate of R_i with a concentration of C_i and flows out at a rate of R_o with a concentration of $C_o(t)$. Two equations we can immediately come up with are

$$V(t) = V_0 + (R_i - R_o)t \qquad C_o(t) = \frac{X(t)}{V(t)} = \frac{X(t)}{V_0 + (R_i - R_o)t}$$

Next, we can write an first order differential equation for the amount of solute, $X(t)$.

$$X'(t) = R_iC_i - R_oC_o(t) \Longrightarrow X'(t) + \frac{R_oX(t)}{V_0 + (R_i - R_o)t} = R_iC_i$$

Getting to this point is usually the hard part when solving these questions, and the above equation can be solved by using an integrating factor, where K is a free constant.

$$X(t) = C_i(V_0 + (R_i - R_o)t) + K(V_0 + (R_i - R_o)t)^{\frac{-R_o}{R_i - R_o}}$$

If $R_i = R_o = R$, the integrating factor changes and gives the following solution where K is a free constant.

$$X(t) = V_0C_i + Ke^{-\frac{R}{V_0}t}$$

7.5 A Note on Multiple Solutions

It is important to note that when solving a differential equation the solution may not always be unique. If the derivative is ever undefined at a point, solutions can be sort of ”stitched” together using a piecewise function with a break at the undefined point. For example, both of the following are valid solutions to the equation $y' = 1/x$, $y(1) = 2$:

$$y = \ln|x| + 2 \qquad\qquad y = \begin{cases} \ln|x| + 2 & x > 0 \\ \ln|x| + 3 & x < 0 \end{cases}$$

7.6 Differential Equations Techniques

Throughout this section, a variety of notation will be used. Either $\frac{dy}{dx}$ or y' will be used for regular derivatives, while $\frac{\partial F(x,y)}{\partial x}$ or $F_x(x, y)$ will be used for partial derivatives. Whenever Lagrange notation is used (y'), it will always denote derivatives with respect to x.

7.6.1 Separation of Variables

Given a differential equation in the form

$$y' = F(x)G(y)$$

one can separate the x and y terms and derivatives and take the integral of each side, which can be seen when solving equations such as the logistic differential equation.

$$\frac{dy}{G(y)} = F(x)\,dx \Longrightarrow \int \frac{1}{G(y)}\,dy = \int F(x)\,dx$$

7.6.2 First Order Linear

Given a differential equation in the form

$$y' + P(x)y = Q(x)$$

we can multiply both sides of the equation by the integrating factor $e^{\int P(x)\,dx}$, allowing us to take the integral of both sides by using an inverse product rule on the left.

$$[y' + P(x)y]\,e^{\int P(x)\,dx} = Q(x)e^{\int P(x)\,dx} \Longrightarrow ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx$$

7.6.3 Substitution: y/x (Homogeneous)

Given a differential equation in the form

$$y' = F\left(\frac{y}{x}\right)$$

we can utilize the substitution $v = \frac{y}{x}$ which implies that $y' = v + xv'$. After this, separation of variables can be used.

$$v + xv' = F(v) \Longrightarrow v' = \frac{F(v) - v}{x} \Longrightarrow \int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

7.6.4 Substitution: $ax + by$

Given a differential equation in the form

$$y' = G(ax + by)$$

we can use the substitution $v = ax + by$ which implies that $y' = \frac{1}{b}(v' - a)$. After this, separation of variables can be used.

$$\frac{1}{b}(v' - a) = G(v) \Longrightarrow v' = a + bG(v) \Longrightarrow \int \frac{dv}{a + bG(v)} = \int 1\,dx$$

7.6.5 Substitution: $y^{(1-n)}$ (Bernoulli)

Given a differential equation in the form

$$y' + P(x)y = Q(x)y^n \Longrightarrow y^{-n}y' + P(x)y^{1-n} = Q(x)$$

we can use the substitution $v = y^{1-n}$ which implies that $v' = (1-n)y^{-n}y'$. After this, we are left with a first order linear differential equation.

$$\frac{1}{1-n}v' + P(x)v = Q(x)$$

7.6.6 Exact Equations

Given a differential equation in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \Longrightarrow M(x,y)\,dx + N(x,y)\,dy = 0$$

If there is a function $\Psi(x,y)$ such that $\Psi_x = M$ and $\Psi_y = N$, then we can write our solution as $\Psi(x,y) = C$ for some constant C. The test for an exact equation solution is if $M_y = \Psi_{xy} = \Psi_{yx} = N_x$.

7.6.7 Reducible Second Order

Given a second order differential equation where y is missing, the substitution $p = y'$ can be used, which means $p' = y''$, transforming our equation into one of first order.

$$F(x, y', y'') = 0 \Longrightarrow F(x, p, p') = 0$$

Given a second order differential equation where x is missing, the substitution $p = y'$ can be used, which means $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$, transforming our equation into one of first order.

$$F(y, y', y'') = 0 \Longrightarrow F\left(y, p, \frac{dp}{dy}\right) = 0$$

7.6.8 Constant Coefficient Homogeneous Linear Second Order

Given a differential equation in the form

$$ay'' + by' + cy = 0$$

we call this homogeneous since it is equal to 0 and contains no loose x terms. To solve this, we turn it into the following characteristic polynomial and solve.

$$at^2 + bt + c = 0 \Longrightarrow t = r_1, r_2$$

Thus, the solution can be written as $y = c_1e^{r_1x} + c_2e^{r_2x}$ for free constants c_1 and c_2. This can also be extended to higher order equations as long as they are constant coefficient, homogeneous, and linear.

If some root r is repeated n times, its solutions are written as $(c_1 + c_2x + \ldots + c_nx^{n-1})e^{rx}$. Also, if the characteristic polynomial has roots $r = a \pm bi$ with nonzero imaginary parts, its solutions can be written as $e^{ax}(c_1\cos(bx) + c_2\sin(bx))$.

7.6.9 Cauchy-Euler Equations

In the case that the differential equation is of the form

$$ax^2y'' + bxy' + cy = 0$$

we can apply the substitution $v = \ln x$ which after some manipulation, gives us $y' = \frac{1}{x}\frac{dy}{dv}$ and $y'' = \frac{1}{x^2}\left(\frac{d^2y}{dv^2} - \frac{dy}{dv}\right)$. Plugging these in, we can solve this like above.

$$ax^2 \cdot \frac{1}{x^2}\left(\frac{d^2y}{dv^2} - \frac{dy}{dv}\right) + bx \cdot \frac{1}{x}\frac{dy}{dv} + cy = 0 \Longrightarrow a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0$$

$$\Longrightarrow y = c_1 e^{r_1 v} + c_2 e^{r_2 v} = c_1 x^{r_1} + c_2 x^{r_2}$$

In the case of repeated roots, the solutions are of the form $y = (c_1 + c_2 \ln x)x^r$. If the roots are complex and written as $r = a \pm bi$, the solutions can be written as $x^a(c_1 \cos(b \ln x) + c_2 \sin(b \ln x))$. The same substitution can also be applied to a third degree Cauchy-Euler equation, resulting in the following transformation.

$$ax^3y''' + bx^2y'' + cxy' + dy = 0 \Longrightarrow a\frac{d^3y}{dv^3} + (b-3a)\frac{d^2y}{dv^2} + (c-b+2a)\frac{dy}{dv} + dy = 0$$

7.7 Principle of Superposition

If $y_1, y_2, \ldots, y_n$ are the linearly independent solutions to a homogeneous linear differential equation, then for free constants $c_1, c_2, \ldots, c_n$, so is

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

7.8 The Wronskian

Suppose that n functions $f_1, f_2, \ldots, f_n$ are each $n-1$ times differentiable. Then their Wronskian is the $n \times n$ determinant

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

The result of the determinant has two choices:

1. If $W \equiv 0$, the solutions are linearly dependent.
2. If $W \neq 0$, the solutions are linearly independent.

7.9 Laplace Table

Laplace transforms are very unlikely to appear on a regular test but can be useful to quickly solve certain improper integrals. Remember that the general definition of a Laplace transform is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Function	**Transform**
$f(t)$	$F(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{(n-1)}(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$

$e^{at}f(t),\ a \in \mathbb{R}$	$F(s-a)$
$u(t-a)f(t-a),\ a \in \mathbb{R}^+$	$e^{-as}F(s)$
$f(t) * g(t)$	$F(s)G(s)$
$tf(t)$	$-F'(s)$
$t^n f(t),\ n \in \mathbb{Z}$	$(-1)^n F^{(n)}(s)$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\,d\sigma$
$f(t)$, period p	$\dfrac{1}{1-e^{-sp}}\displaystyle\int_0^p e^{-st}f(t)\,dt$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
$t^n,\ n \in \mathbb{Z}$	$\dfrac{n!}{s^{n+1}}$
$t^a,\ n \in \mathbb{R}^+$	$\dfrac{\Gamma(a+1)}{s^{n+1}}$
$e^{at},\ a \in \mathbb{R}$	$\dfrac{1}{s-a}$
$t^n e^{at},\ n \in \mathbb{Z},\ a \in \mathbb{R}$	$\dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$
$\cos(kt),\ k \in \mathbb{R}$	$\dfrac{s}{s^2+k^2}$
$\sin(kt),\ k \in \mathbb{R}$	$\dfrac{k}{s^2+k^2}$
$\cosh(kt),\ k \in \mathbb{R}$	$\dfrac{s}{s^2-k^2}$
$\sinh(kt),\ k \in \mathbb{R}$	$\dfrac{k}{s^2-k^2}$
$\dfrac{1}{2k^3}(\sin(kt) - kt\cos(kt)),\ k \in \mathbb{R}$	$\dfrac{1}{(s^2+k^2)^2}$
$\dfrac{t}{2k}\sin(kt),\ k \in \mathbb{R}$	$\dfrac{s}{(s^2+k^2)^2}$
$\dfrac{1}{2k}(\sin(kt) + kt\cos(kt)),\ k \in \mathbb{R}$	$\dfrac{s^2}{(s^2+k^2)^2}$
$u(t-a),\ a \in \mathbb{R}^+$	$\dfrac{e^{-as}}{s}$
$\delta(t-a),\ a \in \mathbb{R}^+$	e^{-as}